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# The influence of computers and informatics on mathematics and its teaching

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# THE IMPACT OF SYMBOLIC MATHEMATICAL SYSTEMS ON MATHEMATICS EDUCATION

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Symbolic manipulators, that is, computer programmes with the capability of carrying out symbolic computations, for example, in calculus or linear algebra, are now widely available. While these are well-established tools in many areas of mathematics, science and engineering, it must be recognized that they are still in their infancy with respect to their use in mathematics education. They represent an ineluctable challenge to current approaches to the teaching of mathematics and there is a belief among some members of the mathematical community that electronic information technology, through these symbolic capabilities, will exert a deep influence on how and what mathematics is taught and learned (for example see Page (1990)). However no clear pattern has yet emerged on how such an influence is to be articulated.

This paper will discuss certain aspects of the impact of symbolic manipulators on mathematical education in the upper secondary years and the first few years of university. It is by no means intended to give the final word on such a vast field as much work is in progress and the technical environment (computer hardware/software and calculators) is constantly improving. The aim of this paper is rather to examine some of the major issues and to indicate general trends which have developed since the 1985 ICMI Study on "The Influence of Computers and Informatics on Mathematics and its Teaching". The influence of symbolic manipulators on more advanced (senior) mathematics courses will not be explored. This is not intended to belittle their impact at this level but rather to concentrate on those years where these systems must be implemented in order to benefit the largest possible number of students in mathematics courses. The influence of these systems and their mathematical foundations (see for example Davenport, Siret and Tournier (1988)) will be thrust into the upper level courses by more capable and interested students as they progress through the system.

Section 1 defines Symbolic Mathematical Systems in broad terms and presents an example of their potential use in mathematics education. Section 2 raises some general concerns related to the impact of these systems on mathematics education while Section 3 discusses implementation of some of the required changes in secondary and university mathematics education. The Appendices provide the following additional information: (1) references dealing with the technical aspects of some of the better known Symbolic Mathematical Systems, (2) further illustrations of the capabilities of these systems, and (3) references to current projects aimed at the integration of such systems into mathematics education.

## 1. Symbolic Mathematical Systems

The term Symbolic Mathematical Systems is used to define calculator and microcomputer systems which provide integrated (1) numeric, (2) graphic, and (3) symbolic manipulation capabilities[1]. *Numerical* computations have always been included in the domain of both the calculator and the computer. This capability is usually thought of as the ability of doing decimal arithmetic. For example, if $1/3 + 1/9$ is input, then the approximate solution 0.444444 (to some prespecified number of digits) is provided. Symbolic Mathematical Systems have the ability to perform rational arithmetic, that is, to give the exact answer $4/9$ if the input is $1/3 + 1/9$. The user must request the decimal approximation if it is desired. *Graphing* is a more complicated numerical activity. Calculators with graphic capabilities (for example the Casio fx-7000G, Hewlett-Packard HP-48SX or Texas In-

[1] It should be noted that in a much more general context, the expression "symbolic computation" could be construed as referring to various types of symbolic objects, for example as described by Aspetsberger and Kutzler (1988): geometric objects (*computational geometry*), logic objects (*automatic reasoning*), programmes (*automatic programming*). The concerns of this paper are limited to computations involving algebraic expressions, so that typical topics of the field are symbolic differentiation and integration, calculation of sums and limits in closed form, symbolic solution of systems of equations and of differential equations, polynomial factorization, manipulation of matrices with or without numeric entries, arbitrary precision rational arithmetic computations, etc. These are sometimes misleadingly called "Computer Algebra Systems" — but they can do much more than algebra as will be illustrated by the examples in this article.

struments TI-81) as well as microcomputer graphing programmes are available. To many mathematicians and mathematics educators ***symbol manipulation*** by calculators (for example the Hewlett-Packard HP-28S or HP-48SX) and microcomputer programmes (for example Maple, Mathematica, Derive to name a few) was a most unexpected development. It is the one capability which has the potential of producing the most radical changes in the teaching of mathematics at the secondary school and university levels.

To convey a feeling for some of the capabilities of Symbolic Mathematical Systems and how they could be used in a calculus class, consider the following example of a session with a specific system (namely Maple but this particular choice is not crucial). Such an example could be done in class, or could be structured as part of a laboratory exercise. The example illustrates the numeric, graphic and symbolic manipulation capabilities of the system and shows the system can be used in a mode which requires no programming by the user, but only the knowledge of a few command words. For ease of understanding lines starting with a # (and in *italics*) are external comments, lines starting with a ::: are the user's input and the lines in **bold** are the (Maple) system's response.

*#The task is to explore the derivative of ln(x) using*
*#the definition of the derivative. First the limit of*
*# (ln(t)−ln(4))/(t−4), called y, as t approaches the*
*# integer value 4 is explored from a numerical*
*#point of view, by computing the value of y around*
*#t = 4. Clearly the value at t = 4 does not exist.*

::: y:= (ln(t)−ln(4))/(t−4);

$$\mathbf{y := \frac{\ln(t) - \ln(4)}{t - 4}}$$

*#At t = 3.99*

::: subs(t=3.99,y);

$$\mathbf{-100\ \ln(3.99) + 100\ \ln(4)}$$

*#Evaluation using floating-point arithmetic of this*
*#last displayed expression then gives*

::: evalf(");

**.250313**

*#At t = 3.999*

::: evalf(subs(t=3.999,y));

**.250031**

*#At t = 4.01*

::: evalf(subs(t=4.01,y));

**.249688**

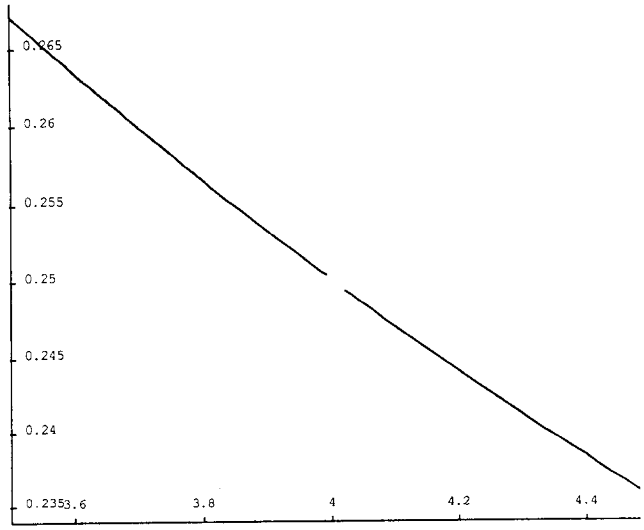
*#At t = 4.001*

::: evalf(subs(t=4.001,y));

**.249969**

*#Looks as though the function is approaching 0.25*
*#as t approaches 4. Does the graph support this?*
*#A plot of y for 3.5 < t < 4.5 is obtained.*

::: plot(y,3.5..4.5);




*#Yes it does and the graph indicates by a hole that*
*#the function is not defined at t = 4, where y is*
*#approximately equal to 0.25. One repeats this*
*#experimentation with a few more integer and*
*#rational cases, for example 5, 3/2, 7/3. Then the*
*#symbol manipulation capabilities can be used to*
*#evaluate the limit directly,*

::: limit((ln(t)−ln(3/2))/(t−3/2),t=3/2);

$$\mathbf{\frac{2}{3}}$$

*#suggesting that the limit of*
*#(ln(t)−ln(a))/(t−a) as t approaches a is 1/a*
*#for all a>0. This is confirmed by the system.*

::: limit((ln(t)−ln(a))/(t−a),t=a);

$$\mathbf{\frac{1}{a}}$$

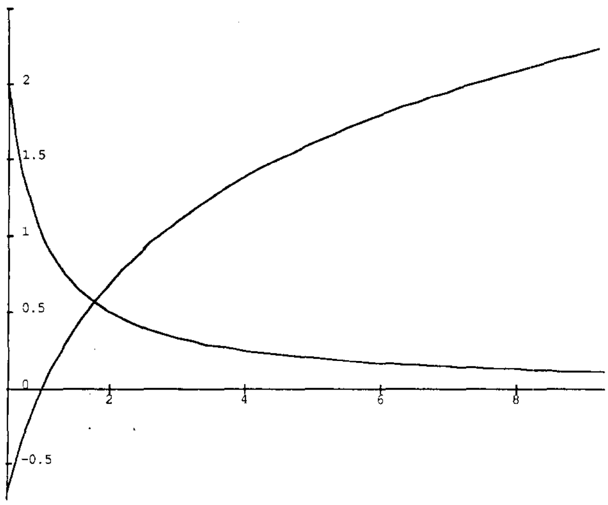
*#Which is also confirmed by the differentiation*
*#capability of the system.*

::: diff(ln(a),a);

$$\mathbf{\frac{1}{a}}$$

*#Does the derivative have the properties expected?*
*#Plot the function and its derivative on the same*
*#graph (in some judiciously chosen interval !).*

::: plot({ln(t), 1/t}, 0.5..10.5);




*#Yes ln(t) is a monotonically increasing*
*#function and the derivative is shown to be*
*#positive in the chosen range. Measurements along*
*#the axes appear to confirm the previously*
*#computed points (4,1/4) etc. The formal*
*#definition of the limit can also be explored, namely,*
*#L is the limit of y as t tends to a if for every*
*#eps > 0 there is a del > 0 such that if*
*#0 < |t−a| < del then |y − L| < eps.*
*#Consider the case explored earlier where it was*
*#conjectured that the limit of y as t tends to 4 is*
*#1/4. Select eps = 0.01 > 0; is there a del such*
*#that for 0 < |t − 4| < del, then |y − 1/4| <*
*#0.01? The condition |y − 1/4| < 0.01 can be*
*#rewritten as 1/4−0.01 < y < 1/4+0.01.*
*#This is solved using the system. (Maple solves*
*#different types of equations: algebraic, numeric,*
*#differential, etc.) In this case we are*
*#interested in the numerical solution of an*
*#equation in one variable. Numerical procedures*
*#for the solution of such equations often require*
*#the user to specify an interval within which one*
*#expects to locate the root. In this particular case*
*#Maple does not require such a prompt and*
*#provides the following:*

::: fsolve(y=0.24,t);

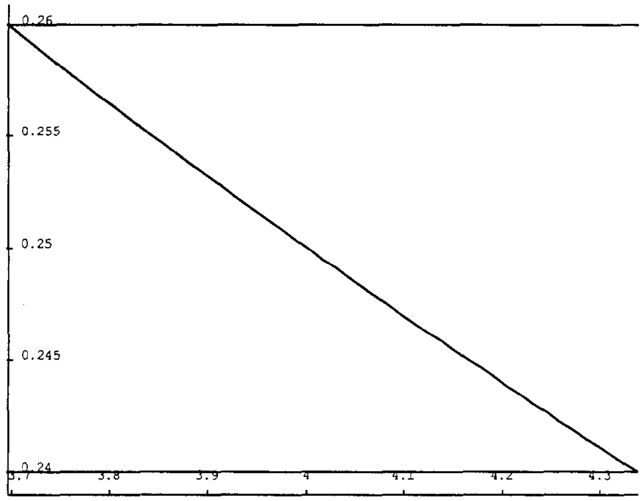
**4.33789986**

::: fsolve(y=0.26,t);

**3.696303966**

*#From these two values it is concluded (based on*
*#the continuity of the log function) that there is a*
*#del, for example 0.2, such that when*
*#0 < |t − 4| < 0.2, then y is in the specified*
*#range. This is visualized with the following plot,*
*#where we notice that the graph of y is*
*#completely contained in the specified window.*

::: plot({y, 0.24, 0.26}, 3.6963..4.3379);




*#To demonstrate how incredibly sensitive and*
*#accurate the limit procedure is, one can consider*
*#the following.*

::: limit((ln(t)−ln(3.2))/(t−3.2),t=3.2);

**undefined**

*#What happened? To resolve this apparent*
*#anomaly the user must realize that elementary*
*#functions involving numbers other than integers*
*#or rationals are approximated (the calculator*
*#mode), that is ln(3.2) is evaluated as shown by*
*#the following output*

::: y:=(ln(t)−ln(3.2))/(t−3.2);

$$\mathbf{y := \frac{ln(t) - 1.163150810}{t - 3.2}}$$

*#and, because of the numerical approximation, the*
*#limit of y as t approaches 3.2 does not exist.*

For those who are not familiar with Symbolic Mathematical Systems Appendix 2 provides further examples of their capabilities. While special purpose packages have been created to cover specific aspects or topics within the mathematics curriculum, this paper is concerned with "full service" Symbolic Mathematical Systems which can become part of mathematics education across different courses and at different levels. The more powerful systems were originally created to help individuals perform complicated yet algebraically routine mathematics. There is no evidence that the introduction of inexperienced students to more dedicated (smaller specially developed systems addressing one part of the syllabus) has been more successful than introducing them to the larger more sophisticated systems.

## 2. Mathematics Education Concerns

Mathematics educators must continually make decisions about what mathematics is to be taught, how it is to be presented and what student activities are to be required or encouraged. To this decision making must now be added the role of Symbolic Mathematical Systems. These systems are a fact of life and can no longer be ignored. Mathematics educators have the responsibility to decide consciously whether this environment is to be included within the student's educational experience and what should be the exact role of the Symbolic Mathematical System. This decision cannot be taken lightly for these systems can perform all the mathematical techniques presently included in secondary school mathematics programmes and most of those included in the first two years of university mathematics. The decision to include or exclude the experience of a Symbolic Mathematical System has far reaching implications to the *student*, the *teacher* and to the *curriculum*. These are now considered in turn.

### a) Implications to the student

The magnitude of the experiences promised to the *student* by Symbolic Mathematical Systems is illustrated by the following allegory:

A person explores her surroundings by walking (pencil and paper) — many interesting things are discovered, but situations in the neighbouring province are too far away to be experienced, so the use of a car (standard scientific calculator) is allowed. As she drives along, local attractions are overlooked in order to get to her destination. However, even with this mode of transport, she cannot explore distant lands, so an airplane (Symbolic Mathematical System) is provided. She lands in a country where the language is not her own, customs are different — as educators we would try to prepare her for this shock — but there is nothing that is quite like being there. What potential benefit awaits her! — she can now explore concepts which were unknown before and she can contrast, compare and have a different view and appreciation of her own culture and home environment. In this new land she continues to use the other modes of transportation, namely, walking and driving to enhance her experience.

Mathematics education has many of the properties of this allegory. Individuals develop their mathematical understanding in various ways. Due to the different roles played by the left and right hemispheres of the brain, it is most likely that the representation of mathematical concepts in complementary modes such as numeric, graphic, and symbolic will enhance the learning process. For the first time in the history of mathematics education Symbolic Mathematical Systems offer the ability to move easily and rapidly between these different representations. It is expected that the use of paper and pencil will be retained by most students; however, one should not be surprised to find students who can operate completely within the computer environment since most systems now provide for easy interplay between word processing and Symbolic Mathematical Systems.

### b) Implications to the teacher

For the *teacher* Symbolic Mathematical Systems are remarkable not only because they can be used to directly perform rational, symbolic or graphic computations but, more importantly, because of what they suggest about mathematics itself and about mathematics teaching. As Young (1986) puts it, "(...) we are participating in a revolution in mathematics as profound as the introduction of Arabic numerals into Europe, or the invention of the calculus. Those earlier revolutions had common features: hard problems became easy, and solvable not only by an intellectual elite but by a multitude of people without special mathematical talents; problems arose that had not been previously visualized, and their solutions changed the entire level of the field." Symbolic Mathematical Systems are part of this revolution. They can serve to help concept development and, by permitting easy and efficient processing of non-trivial examples, they can stimulate exploration and search for patterns[2], generalizations or counter-examples. The teacher must now question the whole of mathematics education. For example, it is increasingly difficult to justify wanting students to become good symbol manipulators unless it can be shown that such procedural skills are essential to an understanding of the underlying mathematical concepts — but no one has yet so shown. However this does not imply that students

[2] "The rapid growth of computing and applications has helped cross-fertilize the mathematical sciences, yielding an unprecedented abundance of new methods, theories and models. (...) No longer just the study of number and space, mathematical science has become the science of patterns, with theory built on relations among patterns and on applications derived from the fit between pattern and observation." Steen (1988).

no longer need develop "symbol sense" —just as the arithmetic calculator has not reduced the need for "number sense". Suddenly the teacher is brought to question both the *content* of the mathematics courses and their *presentation*. As the former relates to curriculum more directly, the latter concern is addressed first.

The teacher must consider many factors which affect the learning of mathematics. An important factor is the social environment. Some students find it easier and more enjoyable to work on their own while others prefer to work in groups. Some depend on the verbal or written or visual presentation of mathematical concepts by those who understand them. Others find this distracting and prefer to work directly from books. Computers provide opportunities to enhance these social environments. They also introduce a new factor — the computer — which may, for some individuals, erect new barriers and difficulties. It is therefore important for mathematics educators to provide alternative environments for students to experience. Individuals will then be in a position to evaluate them and decide which provide the most opportunities for the development of their mathematical knowledge.

Symbolic Mathematical Systems can be integrated into mathematics education in a number of different ways. The three most obvious ones are:

(1) The teacher can use it as part of a *lecture* or *class presentation*. This requires some projection facilities to allow the students to see what appears on the computer screen. For the mathematics instructor the use of such a system in the classroom provides very different class dynamics. Attention has to be paid to typing, errors, unexpected forms of expressions, graphs which appear different from the traditional book presentation (cf. Muller (1992)), multiple answers, etc. Many mathematics instructors find this situation difficult to handle. Perhaps the central aspect in the successful integration of a Symbolic Mathematical System in the classroom is a necessary evolution of the role of the teacher where intervention is no longer restricted to exposition. Instead the teacher must become a "facilitator" creating a context appropriate for a fruitful interaction between the student, the machine and the mathematical concept. The lecture-examples format must be replaced by a more open-ended approach. Although such a point of view is desirable even in a computer-free classroom, it becomes essential when computers come into play. One of the reasons why films and videos have played such a small role in the mathematics classroom may be the mathematician's belief that you understand mathematics by doing it and not by viewing it. Unlike film, Symbolic Mathematical Systems provide an active environment requiring constant intervention and change of direction. Nevertheless it would be naive not to realize that many teachers will find the sacrifice of traditional security quite threatening. This will be especially true of mathematics teachers who see their role as one of "professing" well-polished mathematical knowledge. White (1989) has suggested that the use of Symbolic Mathematical Systems "can be assimilated most easily in traditional teaching methods and curricula." However, in practice, finding an appropriate role for the teacher may prove to be a major barrier for the universal introduction of Symbolic Mathematical Systems into the traditional lecture presentation and teachers should seriously look at alternative and/or complementary modes of implementation. Even though introducing an occasional Symbolic Mathematical System demonstration into a traditional set of lectures is a start, what is needed is a complete rethinking of the objectives of those lectures.

(2) The technology can also be used in scheduled *laboratory sessions*. This is probably the least threatening mode of introduction for the teacher. Laboratory activities can be developed and tested before the students try them. Students can be given materials to prepare for the laboratory sessions and support can be provided for the students during their scheduled laboratories. The physical laboratory setup can vary. There are advantages to having students working with their own system and advantages to having four to six students working together with a single system. Activities appropriate for laboratory work with a Symbolic Mathematical System should not be a simple duplication of activities which can be achieved just as easily with pencil and paper. What are appropriate activities? Clearly the lack of sustained experience limits one's vision. Nevertheless it is suggested (cf. Muller (1991)) that laboratory activities should meet one or more of the following general attributes:

(a) they encourage exploration of mathematical concepts;
(b) they probe inductive reasoning and/or pattern recognition;
(c) they investigate interrelationships between different representations - algebraic, graphical, numerical, etc.;
(d) they involve problems which would be very difficult and/or too time consuming to solve without the technology.

One can visualize a situation where the *lecture* and *laboratory* activities are merged and the lecture presentation takes place in an area where students have access to systems. Because students work at different rates with systems it is quite a challenge to lecture in the traditional way and have students working independently or in groups. The lecture dynamics parallels the situation where one allows time for students to work independently on problems. Devitt (1990) and others have used this method.

(3) It is important to prepare for the time when students will have easy *individual access* to Symbolic Mathematical Systems. A consequence of technological improvements is that a calculator with integrated numeric, symbolic and graphic capabilities is no longer a dream and that such devices can only become progressively more powerful and cheaper. Furthermore, one can expect that the difference between portable computers and calculators will become less apparent. Denying the use of such calculators/computers in structured mathematics instruction does not solve the problem of their existence, and their access by a few more fortunate students. Every society believes that its students should be exposed to all environments which promise a richer educational experience. Of course many situations arise where that society cannot afford to provide a particular environment. Nevertheless this does not relieve teachers from their responsibility to make every possible effort to provide them.

#### c) Implications to the curriculum

There is no doubt that Symbolic Mathematical Systems will have impact on the *curriculum*. What is in question is the magnitude of this impact. There is already evidence that traditional courses will have to change if these systems are to be integrated in any meaningful way. Even with relation to elementary concepts such as graphing, Dick and Musser (1990) observe: "This change in approach made possible by these calculators marks a significant shift in how graphing could be perceived by students. Instead of as a final task to be completed, graphing can assume the role as a problem-solving heuristic and a tool for exploration." Thus the traditional calculus approach of finding what the graph looks like is turned around to using calculus and numerical methods for locating more accurately the properties which are known to exist. Students rapidly come to appreciate both the exactness of non-numerical algebra and the approximation techniques underlying numerical analysis.

The decision as to what extent Symbolic Mathematical Systems are to be included in the mathematics curriculum will vary according to the groups of students being considered and their level. For instance, one could have requirements for a student in a university mathematics programme different from those for a student registered in a mathematics service course. In this respect there is much evidence that shows that scientists from other disciplines (see for example Lance et al. (1986)) serviced by mathematics departments are interested that their students not be denied the use of Symbolic Mathematical Systems. Such scientists, often more open-minded than pure mathematicians with respect to technological developments, simply perceive Symbolic Mathematical Systems as tools that can help them in their work and so are eager to use them. It is therefore necessary to reassess the proper balance in the requirement of basic symbolic manipulation skills and in the choice of topics covered in the various mathematics curricula.

Mathematics educators must make sure that in connection with domains where Symbolic Mathematical Systems can play a role, their courses help students acquire the appropriate intellectual skills. The required skills, while not really "new", are very often given little place in most traditional teaching: these are *interpretive* skills, needed to make mathematical judgements, to appreciate the validity and limitations of the tool being used, to assess the reasonableness of the computed "answer" (cf. Hodgson (1990)). Such skills, being much more demanding than traditional algorithmic ones, will require the student to be confronted with a substantial number of theoretical notions. Thorough understanding of mathematical concepts is thus now surely as — or even more — necessary in mathematics education as it has ever been (cf. Hodgson (1987)).

Another issue which is important in a pedagogical context is the extent to which the symbolic package will act in a "black box" mode or on the contrary give indications about how the "answer" to a particular problem can be obtained. A White-Box/Black-Box Principle has been advocated by Buchberger (1990) in relation to the question: Should students learn integration rules? Buchberger's point of view is essentially that in a stage where a certain mathematical topic is being learned by the student, the use of a Symbolic Mathematical System realizing the pertinent algorithms as "black boxes" would be a disaster. So he calls for systems that would feature the possibility to use an algorithm both as a "black box" (as is most often the case with existing systems) and as a "white box", i.e. in a step-by-step mode in which the reduction of the problem to subproblems is exhibited and in which the user could eventually interfere. A similar view is taken by Mascarello and Winkelmann (1992) in this volume. They claim that even if all the details of the internal functioning of a Symbolic Mathematical System are not usually essential to users, they must not remain totally hidden: understanding of main ideas and fundamental restrictions are necessary for proper use of (what they now call) the "grey" boxes.

Once these systems have been introduced into the mathematics courses then the student evaluation must change to reflect this new environment. As less emphasis is placed on certain techniques and more time is spent on concepts, the testing procedures must also change. Osborne (1990), Beckmann (1991) and others have started to address this issue.

It is clear that much experimentation and research are needed to establish how best to use Symbolic Mathematical Systems in the different courses, with the wide-ranging mathematical capabilities of students, and with the various attitudes of teachers. Appendix 3 provides a list of ongoing projects which are addressing some of these concerns.

## 3. Effecting curriculum changes

Generally curriculum changes in the secondary school system require much time to be implemented but when they happen, they are universally implemented: this is a direct consequence of the highly centralized administration of secondary school programmes in almost all educational systems. On the other hand curriculum changes in university courses can be far more spontaneous, but they tend to be localized to a particular course or section of a course, usually under the commitment of one or a few highly motivated individuals. Therefore the introduction of Symbolic Mathematical Systems in secondary school and university mathematics education poses problems of a different nature. In the former, to affect curriculum change one must convince a small group of influential curriculum makers. For the latter, to ensure that the use of Symbolic Mathematics Systems becomes integrated in courses, it is necessary to expose the majority of faculty members of the department to these systems. Kozma's (1985) study on instructional innovation in higher education supports this view. He contrasted projects which were collaboratively developed with those developed by individuals and found that the former were much more likely to be institutionalized. This section discusses some of the time and effort consuming activities which are required when introducing Symbolic Mathematical Systems in both upper secondary and university mathematics education.

The number of different Symbolic Mathematical Systems is expanding rapidly. Some of them have even been developed specifically for education at secondary school or at the beginning of university education. In most systems, especially the more recent ones, attention is being paid to make them more user friendly, that is, easier to use. A list of references which review some of the better known systems is provided in Appendix 1. While the choice of a specific Symbolic Mathematical System appropriate for use in a given classroom context might rest on various criteria (e.g. hardware facilities, level of instruction, topics to be covered, etc.), it is clear that some basic requirements must be met by those systems. For instance the use of the software should be transparent, that is students should spend their time thinking about the mathematics, and not how to operate the computer. Documentation should be essentially unnecessary for users, so that what needs to be done at any point should be apparent (some on-line "help" facility might however be useful in this respect). The software should be robust so that students' (sometimes unpredictable) behaviour should not cause it to crash or hang up too easily. It should interact easily with some word-processor, either internally to allow preparation of "notebooks" integrating word-processed text inserted in the middle of active symbolic software code, or externally to facilitate preparation of reports by students. But most important of all the program, whether used in a tutorial or interactive mode, should be devised so as not to foster the myth of computer omniscience and infallibility too often rooted in students' minds: while the computer brings in speed and reliability, it is the human being who has the intelligence and the ability to reason and make decisions.

As the cost of basic microcomputer technology continues to drop, one would hope for an analogous reduction in the price of hardware necessary for supporting Symbolic Mathematical Systems. While this has happened in some cases, this is not the general rule. Indeed, one should be aware that the general software development trend has been to demand more and more memory and disk space, thereby requiring more powerful and more expensive microcomputer units. Software developers tend to think in terms of the latest available (or forthcoming) hardware facilities, and experienced users call for more integration, namely word-processing, symbol and graphic manipulation, spreadsheets, etc., all of which push up the requirements of the computer system. Thus the implementation of curriculum change involving Symbolic Mathematical Systems requires financial planning for the purchase of equipment and software. Budgets must also be allocated to the maintenance of both hardware and software. Mathematics departments generally have little experience in requesting monies. This has tended to be the prerogative of Science, Physical Education, Fine Arts and other departments. In secondary schools funds are sometimes allocated for implementation of curriculum changes, but these are unlikely to be sufficient. In a university setting it may be worthwhile to run experimental sections to accumulate evidence of improvements in traditional indicators and to obtain faculty and student attitudes and responses to these systems.

In a context where a lot of importance is given in the literature to various symbolic software running on microcomputers, it might be tempting to overlook calculator technology. But the calculator is not restricted to school applications or to computations on numbers in so-called "scientific" notation! There are a number of calculator projects reported in the literature (see for example Nievergelt (1987) and Demana and Waits (1990)). It is true though that present calculators only have limited graphic and symbolic manipulation capabilities. But developments in electronic technology strongly suggest that such more powerful and user-friendly calculators will most certainly be a reality in a not too distant future. To equip a class or for individual use, calculator technology should thus be seriously considered. This is especially true in situations where, for instance, electricity supplies tend to be unreliable.

Once the equipment (hardware/software) has been purchased, meaningful mathematics activities for the students must be developed. Few such activities are available, although some recent publications provide examples in calculus: see for example the Mathematical Association of America Notes Series (P8) and the Maple Workbook (Geddes et al. (1988)) referenced in the Bibliography. But redefining objectives for a course or building pertinent activities is a daunting task. And for such a quest to have a lasting effect, it should be undertaken not by one individual (with eventual loss of the effect, should that individual be away for a while), but rather by a group, for instance by a majority of the faculty members within a mathematics department. This raises the difficult question of how to react to a possible lack of interest by some of those faculty members. After all, most are busy people and are not willing to invest large amounts of their limited time unless there is some evidence that the result will be worthwhile. This is even more true when students' attitudes towards the use of Symbolic Mathematical Systems in the classroom are not as positive as what could have been expected (see for instance Muller (1991) for an attitudinal survey of some teaching experience with a Symbolic Mathematical System).

The principal word of warning is certainly that implementing the necessary curriculum changes takes a lot of human resources in the form of time and dedication. It takes time to conceive the "new course", to develop meaningful students activities, to prepare new materials, to devise tools for assessment. And this must be done in contexts where often no (or little) credit is given to those who embark on such a task! Furthermore released time, supervision, hardware and software all require financial resources in an area where administrators have not been used to allocating funds. Mathematics educators must convince school or university administrations and funding bodies that such an investment is essential and is worth its value! And what is needed to support the argument is a critical analysis of controlled experiments, rather than anecdotal reporting of experiences.

## 4. Conclusion

The introduction of Symbolic Mathematical Systems into mathematics programmes should be considered within the broader context of the impact of technology on mathematics education. Mathematics teachers who have successfully integrated other software into their teaching of geometry, statistics etc. as well as computer scientists can offer useful insights and pedagogical points of view. Most of the projects aimed at the integration of Symbolic Mathematical Systems into mathematics teaching are either still under way or, if concluded, have results which are difficult to interpret. For example, how does one separate the effects of a Symbolic Mathematical System from other effects, such as those generated by the enthusiam of those involved with the experiment or the effects produced by the availability of additional resources? It is most probably too early to look for a significant impact on the curriculum (measured by the proportion of students in mathematics courses affected by the existence of Symbolic Mathematical Systems). It appears to be the consensus of those who are using these systems in their teaching that the course is taught differently but that it retains a fairly traditional content.

Thus there are few proposals of changes in the curriculum narrowly defined by course content. Some examples of proposals for change are: Tall (1985,1991) proposes a much greater visual component to calculus teaching; Möller (1990) suggests that the conceptual approach to calculus using

"Lipschitz-restricted" concepts of limit, continuity, differentiation and integration is a much more natural one for students and one in which Symbolic Mathematical Systems are easily integrated; Heid (1988) reports experiments in the resequencing of skills in introductory algebra and calculus where attention to hand manipulation skills was drastically reduced; Artigue et al. (1988) traces the influence of computers on the evolution of the teaching of differential equations; the texts of Hubbard and West (1991) and of Koçak (1989) support this evolution and emphasize the importance of visualization in the study of differential equations.

It is anticipated that many more such experiments will be reported in the near future as there are many projects on the way. Appendix 3 lists some of these projects for which information could be found. Ralston has constantly advocated curriculum reform at all levels of Mathematics Education in order to reflect the reality of today's technology and prepare individuals for future technology; in Ralston (1990), he proposes a framework for the school mathematics curriculum in 2000 which is highly dependent on the use of technology. Yet teachers receive their mathematics education from university mathematics courses in which they make very little use (if any!) of technology. How then can they be expected to realize the importance of technology in Mathematics Education? The reform must be spearheaded by the universities where there exists a greater latitude for experimentation.

There is as yet little evidence that Symbolic Mathematical Systems have had a significant impact on the mathematics curriculum of secondary schools and universities. It appears that the dominant reason for this lack of impetus on the curriculum is the education of teachers and faculty, that is, the lack of experience in these systems by a large proportion of mathematicians. In the university setting there is no evidence to suggest that changes implemented by an individual in one section of a course will have any impact on the course as a whole unless special effort is directed toward involving the majority of the faculty in a department. There are too many interests riding on the required introductory mathematics courses to expect that innovative changes made by one individual will be able to permeate the programme without the support from the majority of individuals in that department.

In spite of the human and financial costs involved, there is no doubt that Symbolic Mathematical Systems must be introduced into the mathematics curriculum. They probably constitute the single most powerful force compelling change in secondary and university mathematics education in the near future. They offer unprecedented opportunities to deepen and revitalize mathematics courses, focusing more on concepts and ideas than on mechanical calculations. While it is true that Symbolic Mathematical Systems, whether on microcomputers or on hand-held calculators, can only become more powerful, more user-friendly and more widely available, they offer right now an exceptional potential for progress in the teaching of mathematics and there is no reason for mathematics educators to delay becoming seriously involved with them. For such an evolution to happen, experiments must be performed on a very large scale and results must be evaluated and widely disseminated.

Appendix 3 contains a (partial) list of projects presently underway, in which Symbolic Mathematical Systems are being used in the classroom both at university and secondary school level. Hopefully these projects can stimulate more mathematics educators to involve Symbolic Mathematical Systems in their daily teaching.

## Bibliography

Since the present volume updates the work and publications of the 1985 ICMI Study on "The Influence of Computers and Informatics on Mathematics and its Teaching" held in Strasbourg, this bibliography is restricted to references which have appeared since that meeting.

### a) Proceedings

There are a number of conference proceedings, books of invited papers and series which provide an overview of classroom and/or laboratory projects and raise philosophical and cognitive issues of using a Symbolic Mathematical System in mathematics education. References P1 and P2 are the outcomes of the 1985 ICMI Study.

P1) The Influence of Computers and Informatics on Mathematics and its Teaching. Supporting Papers of the ICMI Symposium, IREM, Université Louis-Pasteur, Strasbourg, 1985.

P2) Howson, A.G. and Kahane, J.-P. (eds.), The Influence of Computers and Informatics on Mathematics and its Teaching. (Proceedings of the ICMI Symposium, Strasbourg, 1985). Cambridge University Press, 1986.

P3) Johnson, D.C. and Lovis, F. (eds.), Informatics and the Teaching of Mathematics. (Proceedings of the IFIP TC 3/WG 3.1 Working Conference, Sofia, 1987). North-Holland, 1987.

P4) Banchoff, T.F. et al. (eds.) Educational Computing in Mathematics. (Proceedings of ECM/87, Rome, 1987). North-Holland, 1988.

P5) Demana, F., Waits, B.K. and Harvey, J. (eds.) Proceedings of the Annual Conference on Technology in Collegiate Mathematics. (1st: 1988; 2nd: 1989; 3rd: 1990). Addison-Wesley, 1990, 1991.

P6) Cooney, T.J. and Hirsch, C.R. (eds.) Teaching and Learning Mathematics in the 1990s. (1990 Yearbook). National Council of Teachers of Mathematics, 1990.

P7) Dubinsky, E. and Fraser, R. (eds.) Computers and the Teaching of Mathematics: A World View. (Selected papers from ICME-6, Budapest, 1988). Shell Centre for Mathematical Education, University of Nottingham, 1990.

P8) The Mathematical Association of America has issued three volumes in the MAA Notes series related to this field and is preparing a fourth one: a) Smith, D.A. et al. (eds.), Computers and Mathematics: The Use of Computers in Undergraduate Instruction. MAA Notes Number 9, 1988. b) Tucker, T.W. (ed.), Priming the Calculus Pump: Innovations and Resources. MAA Notes Number 17, 1990. c) Leinbach, L.C. et al. (eds.), The Laboratory Approach to Teaching Calculus. MAA Notes Number 20, 1991. d) Computer Algebra Systems in Undergraduate Mathematics Education. To appear.

P9) The Notices of the American Mathematical Society feature a regular column under the title Computers and Mathematics (past editor: J. Barwise; current editor: K. Devlin).

### b) Author Bibliography

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## APPENDIX 1

This appendix provides a list of some Symbolic Mathematical Systems software reviews. It is important to realize that it is extremely difficult to evaluate and benchmark this software. Furthermore many of the evaluations do not take into account possible classroom use and the use by neophytes.

a) The *Notices of the American Mathematical Society* (see reference P9 above) have recently included an individual review of most Symbolic Mathematical Systems:

**Vol. 35, 1988**

*The HP-28S brings computations and theory back together in the classroom*, Y. Nievergelt, 799-804.

*Supercalculators on the PC.*, B. Simon and R.M. Wilson, 978-1001.

*Mathematica — A review*, E.A. Herman, 1334-1344. (Also: *Other comments on Mathematica*, 1344-1349.)

**Vol. 36, 1989**

*MicroCalc 4.0*, G. Gripenberg, 680.

*The menu with the college education (A review of Derive)*, E.L. Grinberg, 838-842.

*Milo: The math processor for the Macintosh*, R.F. Smith, 987-991.

*Milo*, Sha Xin Wei, 991-995.

*PowerMath II*, Y. Nagel, 1204-1206.

*More on PowerMath II*, P. Miles, 1206-1207.

**Vol. 37, 1990**

*Review of PC-Macsyma*, Y. Nagel, 11-14.

*Review of True Basic, Inc. Calculus 3.0*, J.R. Moschovakis, Y. Matsubara, G.B. White, 129-131.

*Derive as a precalculus assistant*, P. Miles, 275-276.

*The right stuff*, K. Devlin, 417-425.

*Almost no stuff in, wrong stuff out*, J.D. Child, 425-426.

*Four computer mathematical environments*, B. Simon, 861-868.

**Vol. 38, 1991**

*Crimes and misdemeanors in the computer algebra trade*, D.R. Stoutemyer, 778-785.

*Periodic knots and Maple*, C. Livingston, 785-788.

b) Other reviews are:

*Symbolic manipulation programs for the personal computer*, K.R. Foster and H.H. Bau, *Science*, 243, 679-684 (1989).

*Derive: A mathematical assistant*, E.A. Herman, *Amer. Math. Monthly*, 96, 948-958 (1989).

*Mathematica: A system for doing mathematics by computer*, L.S. Kroll, *Amer. Math. Monthly*, 96, 855-861 (1989).

*Math without tears*, C. Seiter, *MacWorld* 8 (1), 159-165 (1990).

*Theorist*, J. Rizzo, *MacUser*, 6 (6), 57-59 (1990).

*Mathematica: A system for doing mathematics by computer*, A. Hoenig, *Math. Intelligencer*, 12 (2), 69-74 (1990).

*Theorist*, F. Wattenberg, *Amer. Math. Monthly*, 98, 455-460 (1991).

*Review of Maple in the teaching of calculus*, E.R. Muller, *College Math. J.*, (to appear).

c) Reviews of software and comments on experiments on their use in teaching can also be found in specialized newsletters. Some examples are:

*Computer Algebra Systems in Education Newsletter* published by the Department of Mathematics, Colby College, Waterville, ME 04901, USA.

*Maths & Stats* published by the CTI Centre for Mathematics and Statistics (Computer in Teaching Initiative), Faculty of Education, University of Birmingham, Birmingham, B15 2TT, UK.

*Computer-Algebra Rundbrief* published by Fachgruppe 2.2.1 Computer-Algebra der GI, c/o Dr. F. Schwarz, GMD, Institut F1, Postfach 1240, 5205 St. Augustin, Germany.

## APPENDIX 2

This appendix provides a limited number of examples to illustrate some of the capabilities of Symbolic Mathematical Systems (the system used here is Maple but this particular choice is not crucial). These systems are so powerful that it is impossible to provide a complete overview of their capabilities in a brief text.

*#The system can be used to do some elementary*
*#number theory. For instance the command ifactor*
*#returns the prime factorization of an integer.*

::: ifactor(123456780);

$$(2)^2 \quad (3)^2 \quad (5) \quad (47) \quad (14593)$$

*#With such a tool available, it might be tempting to*
*#venture into some calculations that are not*
*#trivial to do either by hand or in a standard*
*#computer environment. For example the prime*
*#factors of the Mersenne number $2^{67}-1$ were*
*#given in 1903 by F. Cole. It reportedly took him*
*# "three years of Sundays" to complete the*
*#calculations. What can Maple do with that*
*#number?*

::: ifactor(2^67−1);

$$(761838257287)(193707721)$$

*#Done in just a fraction of a minute!! (But*
*#needless to say it is very easy to give as an*
*#input a number that would take "three years of*
*#Sundays" for the system to do.)*
*#Roots of equations can be found directly.*

::: y:= x^3−4*x^2−7*x+10;

$$y := x^3 - 4\ x^2\ - 7\ x + 10$$

::: solve(y=0,x);

$$1, -2, 5$$

*#Even with symbolic coefficients.*

::: z:= a*x^2−2*b*x+c;

$$z := a\ x^2 - 2\ b\ x + c$$

::: solve(z=0,x);

$$1/2\ \frac{2\ b + 2\ (b^2\ - a\ c)^{1/2}}{a},$$
$$1/2\ \frac{2\ b - 2\ (b^2\ - a\ c)^{1/2}}{a}$$

*#The example in Section 1 demonstrated that*
*#these systems are able to compute limits and to*
*#differentiate. They can also find Taylor's Series,*
*#sum finite and infinite series and integrate*
*#and solve differential equations.*

::: taylor(ln(x^2),x=1,4);

$$2(x-1) - (x-1)^2 + 2/3\ (x-1)^3 + O((x-1)^4)$$

::: sum(j^2, j=1..m);

$$1/3\ (m+1)^3 - 1/2\ (m+1)^2 + 1/6\ m + 1/6$$

::: int(x^3*sqrt(x^2−9),x);

$$-3/5\ (x^2-9)^{1/2}x^2 - 54/5\ (x^2-9)^{1/2} + 1/5\ (x^2-9)^{1/2}x^4$$

::: deq:= diff(x(t),t)*t^3+x=0;

$$deq := (\frac{d}{dt}x(t))\ t^3\ + x = 0$$

::: dsolve(deq,x(t));

$$x(t) = \exp(1/2\ \frac{1}{t^2})C$$

*#The next few examples are taken from linear*
*#algebra, namely, the solution of linear*
*#equations and some properties of matrices and*
*#vectors.*

::: eqns:= a*x+b*y=e,c*x+d*y=f; vars:= x,y;

$$\mathbf{eqns:=\{a\ x+b\ y=e, c\ x+d\ y=f\}}$$
$$\mathbf{vars:=\{x,y\}}$$

::: solve(eqns,vars);

$$\{\mathbf{x}=-\frac{\mathbf{b\ f-e\ d}}{\mathbf{a\ d-c\ b}},\quad \mathbf{y}=\frac{\mathbf{a\ f-c\ e}}{\mathbf{a\ d-c\ b}}\}$$

::: A:= array([[1,a,a^2],[1,b,b^2],[1,c,c^2]]);

$$\mathbf{A := array(1..3, 1..3,}\\ \mathbf{[1, a, a^2]}\\ \mathbf{[1, b, b^2]}\\ \mathbf{[1, c, c^2])}$$

::: det(A);

$$\mathbf{b\ c^2 - b^2c - a\ c^2 + a^2c + a\ b^2 - a^2b}$$

::: factor(");

$$\mathbf{-(-\ c + b)(a - c)(a - b)}$$

*#(The symbol " refers to the previously*
*#displayed expression.)*

::: a1:= array([x1,y1,z1]); a2:= array([x2,y2,z2]); a3:= array([x3,y3,z3]);

$$\mathbf{a1 := array(1..3,}\\ \mathbf{[x1, y1, z1])}$$
$$\mathbf{a2 := array(1..3,}\\ \mathbf{[x2, y2, z2])}$$
$$\mathbf{a3 := array(1..3,}\\ \mathbf{[x3, y3, z3])}$$

::: vol:= abs(dotprod(a1,crossprod(a2,a3)));

$$\mathbf{vol :=abs(x1\ (y2\ z3 - z2\ y3)}\\ \mathbf{+\ y1\ (z2\ x3 - x2\ z3)}\\ \mathbf{+\ z1\ (x2\ y3 - y2\ x3))}$$

::: a:= array([[13,5],[5,2]]);

$$\mathbf{a := array(1..2, 1..2,}\\ \mathbf{[13, 5]}\\ \mathbf{[5, 2])}$$

::: c:= eigenvals(a);

$$\mathbf{c := 15/2 + 1/2\ 221^{1/2},\ 15/2 - 1/2\ 221^{1/2}}$$

*#The decimal approximation to these two*
*#eigenvalues gives*

::: evalf(c[1]); evalf(c[2]);

**14.93303438**

**.066965625**

## APPENDIX 3

There is as yet no single source which can provide a comprehensive international listing of projects in the area of Symbolic Mathematical Systems in Mathematics Education. Therefore, the following list cannot be regarded as comprehensive:

1: The Swedish ADM project (*Analysis of the role of the Computer in Mathematics Teaching*); see Björk (1987).

2: The Research Institute for Symbolic Computation at the Johannes Kepler University, Linz, Austria.

3: The Computers in Teaching Initiative Centre for Mathematics and Statistics (*Development of class work sheets to be used with Derive*), see the *Maths & Stats* newsletter published by the CTI Centre, University of Birmingham, UK.

4: *A European Cooperation on the use of Computers in Mathematics*; see Dechamps (1988).

5: The National Science Foundation (U.S.A.) is funding a number of different university projects specifically directed at integrating Symbolic Mathematical Systems into the calculus curriculum. The following is a selection providing a one line statement together with the university and the principal investigator.

*Developing a user friendly interface to Maple and incorporating use of system into teaching calculus*, Rollins College, Winter Park, FL; Douglas Child.

*Developing new calculus curriculum using Maple on a VAX*, Rensselear Polytechnic Institute, Troy, NY; William Boyce.

*Developing a computerized tutor and computational aid based on Maple*, University of Rhode Island, Kingston, RI; Edmund Lamagna.

*Developing an electronically delivered course using the Notebooks feature of Mathematica*, University of Illinois, Urbana, IL; Jerry Uhl.

*Developing a new calculus course emphasing applications and using Mathematica*, University of Iowa, Iowa City, IA; Keith Stroyan.

*Developing a laboratory based calculus course using Mathematica*, Iowa State University, Ames, IA; Elgin Johnston.

*Developing a new calculus course for liberal arts colleges using Mathematica*, Nazareth College, Rochester, NY; Ronald Jorgensen.

*Developing calculus as a laboratory course using MathCad and Derive*, Duke University, Durham, NC; David Smith.

*Emphasizing computer graphics using Maple and emphasizing concepts via programming in ISETL,* Purdue University, West Lafayette, IN; Ed Dubinsky.

*Porting the laboratory calculus developed at Duke over to Mathematica*, Bowdoin College, Brunswick, ME; William Barker.

*Collecting, testing, and desktop publishing the best materials being developed using Mathematica*, University of Michigan at Dearborn, Dearborn, MI; David James.

More detailed informations about projects in the U.S.A. integrating Symbolic Mathematical Systems in the calculus curriculum can be found in the reports contained in reference P8b above: Tucker, T.W. (ed.), *Priming the Calculus Pump: Innovations and Resources.* Mathematical Association of America (MAA Notes Number 17), 1990.