THE MATHEMATICAL EDUCATION OF SCHOOL TEACHERS: A BAKER'S DOZEN OF FERTILE PROBLEMS

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Abstract. In most universities, the mathematical education of pre-university teachers constitutes, among the various tasks of the mathematics department, an important one, at least in terms of the number of students involved. While this has been the case for a long time for secondary school teachers, one can now witness a growing involvement, although still rather modest, of mathematicians in primary school teacher education.

This paper aims at stressing how some of the mathematical themes encountered in teacher education, although elementary, are rich and can lead to gratifying and stimulating mathematical moments. Examples of such themes will be given, pertaining to the mathematical preparation of both primary and secondary school teachers and connected to central topics of the school curriculum. Analyzing elementary mathematics from an advanced point of view can help teachers develop the conceptual understanding of mathematics essential to the soundness of their pedagogical agenda.

1. Introduction

The preparation of mathematics teachers for the primary and secondary school is a multifaceted task, including, besides didactical, psychological or

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practice-based considerations, a substantial component of mathematical knowledge. In most universities, the mathematical education of pre-university teachers constitutes, among the various tasks of the mathematics department, an important one, at least in terms of the number of students involved. While this has been the case for a long time for secondary school teachers, one can now witness a growing involvement, although still somewhat modest, of mathematicians in primary school teacher education. Such an involvement of mathematicians in the mathematical education of pre-university teachers is extremely important, as mathematicians can bring to the student teachers a unique perspective on mathematics itself, as well as on its many links to their future work.

In this paper, I want to give examples of mathematical problems, most connected to central topics of the school curriculum, which I have used successfully over the years with student teachers of both the primary and secondary level. I hope through these problems to illustrate how fertile and far-reaching in terms of mathematical content are some of the topics of interest to school teachers. The mathematical themes thus encountered, although elementary, are rich and can lead, I believe, to gratifying and stimulating mathematical moments for mathematicians themselves. Moreover these problems also nicely show how a given topic can be approached, depending upon the target audience, at a very elementary, or not so elementary, level.

But first I would like to comment briefly on general aspects of the role and responsibilities of mathematicians in the preparation of school teachers. In my paper [20], which can be seen as setting the background for the present one, I have examined the contribution of university mathematicians to the education of teachers. Mathematicians are by no means the sole nor the most important contributors to this complex process, but they do have a major and unique role to play, being thus confronted with a fundamental aspect of their social responsibility. In such an endeavour they are in very good company, as many great mathematicians of the past have devoted much energy to pedagogical issues in general, and to the education of school teachers in particular. Names such as Felix Klein, Henri Poincaré, George Pólya or Hans Freudenthal, to mention just a few, immediately come to mind with respect to this long and important tradition.

A key issue is clearly how supportive the mathematics department is of the involvement, actual or potential, of its faculty in the education of teachers. This concerns among others such (trivial but) crucial aspects as the tenure and promotion scheme of faculty. More generally the basic question is whether such an involvement is regarded by the department as *bona fide* work in mathematics, on the same level, say, as interacting with engineering students or students specializing in mathematics. While various indicators (for instance, peers recognition, promotions, grants, etc.) seem to suggest that teaching mathematics to prospective teachers, and especially to those of the primary school, is regarded by some as low-level work in comparison, for example, with teaching to undergraduate mathematics students or still more to graduate students — not to speak of mathematical research *per se* —, there are signs that mentalities are evolving. Still, the acquisition by active mathematicians of an expertise in matters pertaining to teacher education, and the filling of the "culture gap" between mathematicians and mathematics educators, remain crucial issues in mathematics education in general, and in the education of school teachers in particular.

2. Examples of mathematical topics for teachers: my baker's dozen

The expression *conceptual understanding* is often used to describe the type of mathematical knowledge a teacher needs to develop in order to fully play the role of "facilitator" between pupils and mathematical knowledge. Going much beyond mere factual information, conceptual understanding of mathematics stresses organizing principles and central concepts. It allows teachers to perceive mathematics not as a set of facts to be memorized, but as a coordinated system of ideas.

Quite a few reports have been published over the years, which provide a survey of several researches confirming the importance of a sound teachers' knowledge of mathematical content (see among others Brown and Borko [5], Fenema and Franke [13] or Grossman *et al.* [17]). For instance, a contrast is made in [17] between one teacher with a weak knowledge, who drilled pupils in algorithms to be memorized and applied to predictable problem sets, and another one with a strong mathematical background, who emphasized the "whys" of mathematics and led pupils to think through problems. Analogous considerations can also be found in the famous study of Ma [23], comparing teachers from China and the USA with respect to a "profound understanding of fundamental mathematics", or in the recent work of Ball and Bass [1] on a practice-based "mathematical knowledge for teaching".

As can be expected, the more teachers are comfortable in a deep way with the concepts of mathematics, the better they are equipped to work these concepts with their pupils: "The evidence is beginning to accumulate to support the idea that when a teacher has a conceptual understanding of mathematics, it influences classroom instruction in a positive way." ([13, p. 151]) However, research also suggests that prospective mathematics teachers often lack adequate content knowledge, especially those of the primary level who in addition frequently show high levels of "math anxiety" — see [5, p. 220] and [13, p. 148]. This leads to the difficult question of identifying the type of mathematical experiences prospective teachers should meet during their formal education in universities or colleges.

I will now present problems which I have used successfully over the years with my primary and secondary school student teachers. My main concern in selecting the examples to be worked by teachers is to identify mathematical topics linked to the curricula they will be teaching and with a good potential for providing strong insights. I believe the following examples do meet these expectations and provide stimulating contexts which can invite further exploration by students. I make no claim for high originality here, as most of these problems will probably be already familiar to many of the readers. A point I wish to bring out is the richness of the mathematical phenomena which are accessible in this context: working with prospective teachers can be fully gratifying for the university mathematician, even from the strict point of view of the mathematical problems to be dealt with. I also want to show how some themes are appropriate for both the primary and secondary school teachers, as they can be subjected to an increasing depth of treatment. While many of the following examples could be adapted so to be presented to pupils of the appropriate level, the target audience I have in mind here is the teachers themselves.

In the following, I will use the code **(P)** to indicate a problem that can be worked with primary school teachers — approximately, teachers of grades K–6 —, and **(S)** for a problem mainly appropriate for secondary school teachers — grades 7–12. But such a distinction is necessarily a fuzzy one in some cases.

1. Decimal fractions **(P)**

A decimal fraction is a fraction equivalent to one whose denominator is a power of 10. It thus corresponds to a *terminating* decimal expansion, i.e., an expansion with a finite number of non-zero digits following the decimal mark. For example, $\frac{3}{8} = \frac{375}{1000} = 0.375$.

It is natural to ask for a characterization of the decimal fractions. And the answer is remarkably simple: a fraction a/b (in its lowest terms) is decimal if and only if the denominator is of the form $b = 2^r \cdot 5^s$. The proof involves only the notion of prime factorization and is readily accessible to primary school teachers. It runs as follows. If a/b is decimal, then there is a factor k such that $bk = 10^z$ for a certain exponent z, whence it follows, because of unique factorization, that the only prime factors of b are 2 or 5 (or both). Conversely, for $r > s$ [respectively $r < s$], we can convert a/b into a fraction with denominator 10^r [resp. 10^s] by multiplying its two terms by 5^{r-s} [resp. 2^{s-r}]. For example,

$$\frac{3}{80} = \frac{3}{2^4 \cdot 5} = \frac{3}{2^4 \cdot 5} \cdot \frac{5^3}{5^3} = \frac{375}{10^4} = 0.0375\,.$$

The number of digits in the decimal expansion of a/b is thus given by $\max(r, s)$.

2. Non-decimal fractions **(S)**

If a/b is not a decimal fraction, then its decimal expansion is *non-terminating*: the long division of a by b becomes an infinite process. In such a case however, a certain sequence of digits in the decimal expansion is bound to repeat indefinitely; this follows from the fact that the only possible remainders in the division are $1, 2, \ldots, b-1$. For instance, $\frac{1}{28} = 0.03\overline{571428}$, with the bar indicating the repeating sequence of digits. Such a non-terminating but repeating decimal expansion is called *periodic*. (Non-terminating and non-periodic decimal expansions, such as $0.101001000100001\cdots$, thus correspond to irrational numbers.) In general, a periodic decimal expansion is an expression of the form

$$0.z_1z_2\cdots z_k\overline{z_{k+1}\cdots z_{k+t}}\,,$$

where the integers k and t are chosen to be minimal. The sequence of digits $z_1z_2\cdots z_k$ is called the *pre-period* and $z_{k+1}\cdots z_{k+t}$, the *period*. When $k=0$, the expansion is called *purely periodic*. The following table shows how varied the pattern can be.

		k	t
1/3	$0.\overline{3}$	0	1
1/6	$0.1\overline{6}$	1	1
1/7	$0.\overline{142857}$	0	6
1/11	$0.\overline{09}$	0	2
1/12	$0.08\overline{3}$	2	1
1/13	$0.\overline{076923}$	0	6
1/14	$0.0\overline{714285}$	1	6
1/17	$0.\overline{0588235294117647}$	0	16
1/24	$0.041\overline{6}$	3	1
1/28	$0.03\overline{571428}$	2	6

A problem suitable for secondary school teachers is the following. Let a/b be a fraction given in lowest terms; when a/b is a non-decimal fraction — and we know from example 1 how to decide if this is the case —, can we predict, by simple inspection of the numerator a and denominator b, the length k of the pre-period and the length t of the period of its decimal expansion? The answer to this question is again an application of elementary number theory and can be found in various places (see for instance Niven and Zuckerman [24, pp. 305–308]). But before giving characteristic properties of k and t, I would like to recall the following well-known technique, often introduced in secondary school, for transforming a periodic decimal expansion into a fraction.

Given for instance $x=0.0\overline{18}$ to be written in the form a/b, one first multiplies each side of the equality by appropriate powers of 10 — this amounts to a right-bound move of the decimal mark in the decimal expansion. In this case, we take $10x=0.\overline{18}$ and $1000x=18.\overline{18}$. From the equality $1000x=18+10x$, one obtains $990x=18$, so that $x=1/55$. (A fully rigorous treatment would require the study of convergence of series, but such manipulation of the infinite collection of digits of x is not too disturbing as soon as one accepts non-terminating decimal expansions.)

Let us now get back to our main problem. Given a and b, we thus want to identify minimal values of k and t such that

$$\frac{a}{b}=0.z_1z_2\cdots z_k\overline{z_{k+1}\cdots z_{k+t}}\,.$$

Multiplication by appropriate powers of 10 gives

$$10^k \cdot \frac{a}{b} = z_1 z_2 \cdots z_k . \overline{z_{k+1} \cdots z_{k+t}}$$

and

$$10^{k+t} \cdot \frac{a}{b} = z_1 z_2 \cdots z_k z_{k+1} \cdots z_{k+t} . \overline{z_{k+1} \cdots z_{k+t}} .$$

By subtraction, one finds that

$$\frac{a}{b} \cdot 10^k \cdot (10^t - 1)$$

is a natural number. Let us now write the denominator as $b = 2^r \cdot 5^s \cdot c$, where $\gcd(c, 10) = 1$ (i.e., all the 2's and the 5's have been collected outside of c). Since $\gcd(a, b) = 1$, we can conclude that:

- k is the smallest natural number such that $2^r \cdot 5^s$ divides 10^k — i.e., $k = \max(r, s)$;
- t is the smallest natural number such that c divides $10^t - 1$.

(It is remarkable that these two conditions are fully independent of the numerator a.) We see for instance that a/b is purely periodic ($k = 0$) if and only if $\gcd(b, 10) = 1$. The preceding discussion does not give an explicit formulation for t but it provides us with a method for finding it systematically. A table of the prime factorizations of the numbers $10^t - 1$ for successive values of t is useful in this respect.

$$\begin{array}{rcrcl}
10^1 - 1 & = & 9 & = & 3^2 \\
10^2 - 1 & = & 99 & = & 3^2 \cdot 11 \\
10^3 - 1 & = & 999 & = & 3^3 \cdot 37 \\
10^4 - 1 & = & 9999 & = & 3^2 \cdot 11 \cdot 101 \\
10^5 - 1 & = & 99999 & = & 3^2 \cdot 41 \cdot 271 \\
10^6 - 1 & = & 999999 & = & 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37
\end{array}$$

In using such a table, one must take care to identify the first simultaneous appearance on a given line of all the prime factors of c (with their multiplicity). For instance, since 11 appears for the first time on the second line, the fraction 1/11 has a period of length 2; the same is true of 1/33. The period of 1/27 is of length 3. The fact that 1/7 has a period of length 6 corresponds to 7 appearing for the first time on line 6; 4/21 and 76/77 also have a period of length 6.

The previous table also shows how to construct a fraction satisfying given conditions. Here are, for instance, fractions with $k = 1$ and $t = 3$:

$$\frac{1}{54} = \frac{1}{2 \cdot 3^3}, \qquad \frac{4}{185} = \frac{4}{5 \cdot 37}, \qquad \frac{1}{1110} = \frac{1}{2 \cdot 5 \cdot 3 \cdot 37} .$$

There are many ways of pursuing further this study of the patterns of non-terminating decimal expansions. I now indicate one, which brings into the picture more advanced results from the theory of numbers. It is striking that the periods of 1/7 and 1/17 are optimal, in the sense that all possible remainders are used before repetition — the former has period of length 6 and the latter, 16; such is not the case for $1/3$, $1/11$ or $1/13$, which have "short" periods. How can this phenomenon be explained?

It is not too difficult to show (see Rademacher and Toeplitz [27, Chapter 23]) that in general the length t of the period of a/b is a divisor of $\phi(b)$, the number of positive integers less than or equal to b which are relatively prime to b (that is, Euler ϕ-function). Let us now consider a fraction $1/p$, with p a prime number such that $\gcd(p, 10) = 1$. The length t of its period thus divides $\phi(p) = p - 1$. But for this period to be optimal, one must have $t = p - 1$; and this happens only in exceptional cases. For instance, it can be checked that the only primes below 100 generating optimal periods are 7, 17, 19, 23, 29, 47, 59, 61 and 97. (These are what Conway and Guy [7, p. 161] call "long primes".) When p is such a prime with an optimal period, i.e., when $p-1$ is the smallest positive value of x such that p divides $10^x - 1$, 10 is said to be a *primitive root modulo p*. Very little is known in general about such p's (see Hardy and Wright [18, Section 9.6]). It should be noted that by Fermat's Little Theorem, it is always true that a prime $p \neq 2$ and 5 is a divisor of $10^{p-1} - 1$; the question of optimality of the period of $1/p$ thus amounts to whether there is a smaller value of x such that p divides $10^x - 1$.

Fractions with period of maximal length lend themselves to an entertaining mathematical "trick" which can be used with teachers as an intriguing starting point. Let us consider the following table of multiples of the integer 142 857:

$$\begin{aligned}
142\,857 \times 1 &= 142\,857,\\
142\,857 \times 2 &= 285\,714,\\
142\,857 \times 3 &= 428\,571,\\
142\,857 \times 4 &= 571\,428,\\
142\,857 \times 5 &= 714\,285,\\
142\,857 \times 6 &= 857\,142.
\end{aligned}$$

The pattern of the products is quite striking, as they are all made of the same digits succeeding one another in the very same order. How can this phenomenon be explained?

The answer rests in the sequence of digits "142857" being exactly the period of the fraction $1/7$. The decimal expansions of multiples of $1/7$ is then easily obtained, the digits 1,4,2,8,5,7 coming back again and again in cycles (as is easily seen through hand calculation of $1 \div 7$):

$$\begin{aligned}
\frac{2}{7} &= 0.285\,714\,285\,714\,285\,714\cdots,\\
\frac{3}{7} &= 0.428\,571\,428\,571\,428\,571\cdots,\\
\frac{4}{7} &= 0.571\,428\,571\,428\,571\,428\cdots,\\
&\text{etc.}
\end{aligned}$$

But what about $142\,857 \times 7$? It is then no surprise that this product equals $999\,999$, as one possible decimal expansion for the fraction $7/7$ is

$$0.999\,999\,999\,999\cdots = 1.$$

Further multiples of $142\,857$ can be analyzed in an analogous way.

If one is looking for another integer displaying such a behavior, one has to go up to $588\,235\,294\,117\,647$, which corresponds to the period of the next fraction with maximal length:

$$\frac{1}{17} \quad = \quad 0.\overline{0588235294117647}\,.$$

Further information on periods of fractions can be found in the entertaining paper of Gardner [14]. The interested reader may also wish to examine how to accommodate the results of examples 1 and 2 for bases other than ten — see for instance Rosen [28, Chapter 10].

3. Is the sum of fractions in its lowest terms? (P)

Primary school children are shown that the sum of two fractions can be obtained by transforming them into two fractions with the same denominator and respectively equivalent to the given ones. Hence the general rule

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}\,.$$

But the technique usually proposed to pupils is to use a common denominator as small as possible, in order to minimize the size of numbers appearing in the calculations. This leads to the following *least common denominator algorithm*, which involves the least common multiple of b and d, $\operatorname{lcm}(b,d)$:

$$\frac{a}{b} + \frac{c}{d} \quad = \quad \frac{a \times \dfrac{\operatorname{lcm}(b,d)}{b} \;+\; \dfrac{\operatorname{lcm}(b,d)}{d} \times c}{\operatorname{lcm}(b,d)}\,.$$

For example, one finds with the first method

$$\frac{2}{15} + \frac{7}{9} = \frac{2\cdot 9 + 15\cdot 7}{15\cdot 9} = \frac{123}{135}\,,$$

and with the least common denominator algorithm

$$\frac{2}{15} + \frac{7}{9} = \frac{2\cdot 3 + 5\cdot 7}{45} = \frac{41}{45}\,.$$

It should be noted that in the latter case the sum is directly obtained as a fraction in its lowest terms. A natural question is whether this is always the case: for any a/b and c/d in lowest terms, is the sum obtained by the least common denominator algorithm always a fraction in lowest terms?

That the answer in no in general is easily seen from counter-examples such as $\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$ or $\frac{5}{24} + \frac{1}{8} = \frac{8}{24}$. But what about situations in which $\operatorname{lcm}(b,d)$ is neither b nor d, like in the sum $\frac{2}{15} + \frac{7}{9}$ above? Finding a counter-example in such cases is a nice challenge for primary school teachers.

There is not much interest in looking for a counter-example by simple trial and error. So let us observe the following basic fact: when $\gcd(b,d)=1$, the sum

$$\frac{ad+bc}{bd}$$

is always in lowest terms. The proof of this statement, which involves only elementary properties of the divisibility relation, is readily accessible to a primary school student teacher.

This result gives us a clue about how to find more general counter-examples: use $ad+bc$. For example, we have that

$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6},$$

a fraction in lowest terms. Multiplying each denominator by $ad+bc=5$, we now form the fractions $1/10$ and $1/15$. The least common denominator algorithm gives us

$$\frac{1}{10}+\frac{1}{15}=\frac{5}{30},$$

which is not in lowest terms! This stems simply from the fact that

$$\frac{1}{10}+\frac{1}{15}=\frac{1}{5}\cdot\left(\frac{1}{2}+\frac{1}{3}\right)=\frac{1}{5}\cdot\frac{5}{6}.$$

Other counter-examples can be generated similarly.

4. Manipulating fractions (S)

The following rule for generating infinite sequences of numbers was suggested by John Conway (see [2]). Starting with any two positive numbers, we set

$$x_n=(x_{n-1}+1)\div x_{n-2}.$$

For example, starting with 5 and 8, one obtains, using a calculator,

$$5 \quad 8 \quad 1.8 \quad 0.35 \quad 0.75 \quad 5 \quad 8 \quad 1.8 \quad 0.35 \quad \cdots.$$

Is the periodicity an accident? Let us try another sequence:

$$5 \quad 7 \quad 1.6 \quad 0.37142857 \quad 0.85714285 \quad 5 \quad 7 \quad 1.6 \quad 0.37142857 \quad \cdots.$$

Again a period of length 5. But as this one involves non-terminating decimal expansions, we must be careful that the truncation introduced by the calculator is not misleading us. So let us look at the same sequences, but this time with the terms expressed as fractions:

$$5 \quad 8 \quad \frac{9}{5} \quad \frac{7}{20} \quad \frac{3}{4} \quad 5 \quad 8 \quad \frac{9}{5} \quad \frac{7}{20} \quad \cdots;$$

$$5 \quad 7 \quad \frac{8}{5} \quad \frac{13}{35} \quad \frac{6}{7} \quad 5 \quad 7 \quad \frac{8}{5} \quad \frac{13}{35} \quad \cdots.$$

Aha! Now the rule appears more clearly. We verify it in a general setting by introducing variables a and b. A few calculations give us the sequence

$$a \quad b \quad \frac{b+1}{a} \quad \frac{a+b+1}{ab} \quad \frac{a+1}{b} \quad a \quad b \quad \cdots .$$

Barbeau [2] stresses some nice features of this example. It can be used even with lower grades pupils, thus providing good practice in basic manipulations of fractions (no need for a solution: this is a self-correcting exercise). The general case shows algebra appearing naturally as part of a proof and illustrates the importance of technical skills in keeping control of the process: in order to let the algebraic manipulations stay manageable, one needs to simplify the fractions obtained at each step (this requires performing a factorization at one point).

5. A night at Long Hotel (P)

The following problem is nearly a classic in the mathematics literature for primary school student teachers — see among others the version in Bell *et al.* [3, p. 620]:

> *At Long Hotel, there are n rooms all located along a very long corridor and numbered consecutively from 1 to n. One night, after dinner, the n guests play the following game. The first guest runs down the corridor and opens all the doors. Then the second guest runs down the corridor and closes every second door beginning with door 2. Afterwards, the third guest changes the position of every third door beginning with door 3 (that is, the guest opens the doors that are closed and closes those that are open). In a similar way, the fourth guest changes the position of doors 4, 8, 12, ... This process continues until the nth guest runs down to the end of the corridor to change the position of door n. Which doors are left open and which ones are left closed at the end of the game?*

Although complex at first glance, this problem obeys a remarkably simple rule: *all doors end up closed except those whose numbers are perfect squares.* This stems from the fact that at the end of the process, door d will be open or closed depending only on the parity of the number of divisors of d. The fact that the only numbers with an odd number of divisors are the perfect squares can easily be justified by primary school student teachers.

This problem also provides a nice setting for exploring notions playing an important role in primary school teaching, such as (common) divisor, multiple, gcd or lcm.

6. A night at Circle Hotel (S)

Based on the "what-if-not?" strategy, the following variant of the preceding problem was proposed by Cassidy and Hodgson [6]:

The rooms of Circle Hotel are built around a circular courtyard and are numbered consecutively from 1 to n. One night, after dinner, the n guests want to play the same game as at Long Hotel (see example 5). But since the hotel has the form of a circle, each of the guest could go endlessly round the corridor. So it is agreed that a guest should stop as soon as he or she changes the position of door n — which is bound to happen for every guest. Which doors are left open and which ones are left closed at the end of the game?

This problem is of the sort that responds to an inductive approach. Experimenting with a few concrete cases leads to the following conjecture: *for any n, all the doors are left closed except a single one; and this exceptional door is door n when n is odd, and door $n/2$ when n is even.*

To see why this is the case, let us look at the $n-1$ doors that (apparently) end up closed. Each one of these doors must have been touched by an even number of guests. Taking the set of persons touching a given door, we would like to partition it into subsets, each containing two people, and to do this in a "meaningful manner". It is again instructive to experiment with hotels having a small number of rooms. The following table summarizes what happens in a hotel with 12 rooms. A sign "$\times$" at the intersection of line Gi and column Dj indicates that guest i changes the position of door j sometime during the process. Moreover the index k in the symbol ${}_i\times_k$ gives the order in which guest i touches the doors consecutively.

	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12
G1	${}_1\times_1$	${}_1\times_2$	${}_1\times_3$	${}_1\times_4$	${}_1\times_5$	${}_1\times_6$	${}_1\times_7$	${}_1\times_8$	${}_1\times_9$	${}_1\times_{10}$	${}_1\times_{11}$	${}_1\times_{12}$
G2		${}_2\times_1$		${}_2\times_2$		${}_2\times_3$		${}_2\times_4$		${}_2\times_5$		${}_2\times_6$
G3			${}_3\times_1$			${}_3\times_2$			${}_3\times_3$			${}_3\times_4$
G4				${}_4\times_1$				${}_4\times_2$				${}_4\times_3$
G5	${}_5\times_5$	${}_5\times_{10}$	${}_5\times_3$	${}_5\times_8$	${}_5\times_1$	${}_5\times_6$	${}_5\times_{11}$	${}_5\times_4$	${}_5\times_9$	${}_5\times_2$	${}_5\times_7$	${}_5\times_{12}$
G6						${}_6\times_1$						${}_6\times_2$
G7	${}_7\times_7$	${}_7\times_2$	${}_7\times_9$	${}_7\times_4$	${}_7\times_{11}$	${}_7\times_6$	${}_7\times_1$	${}_7\times_8$	${}_7\times_3$	${}_7\times_{10}$	${}_7\times_5$	${}_7\times_{12}$
G8				${}_8\times_2$				${}_8\times_1$				${}_8\times_3$
G9			${}_9\times_3$			${}_9\times_2$			${}_9\times_1$			${}_9\times_4$
G10		${}_{10}\times_5$		${}_{10}\times_4$		${}_{10}\times_3$		${}_{10}\times_2$		${}_{10}\times_1$		${}_{10}\times_6$
G11	${}_{11}\times_{11}$	${}_{11}\times_{10}$	${}_{11}\times_9$	${}_{11}\times_8$	${}_{11}\times_7$	${}_{11}\times_6$	${}_{11}\times_5$	${}_{11}\times_4$	${}_{11}\times_3$	${}_{11}\times_2$	${}_{11}\times_1$	${}_{11}\times_{12}$
G12												${}_{12}\times_1$

Forgetting the indices and looking only at the signs $\times$ appearing in the table, a striking symmetry is seen: if line G6 is thought as an "axis of reflection", the upper half of the table is reflected in the bottom half, with the exception that line G12 has no symmetrical counterpart (there is also a symmetry with respect to column D6, with the exception of column D12, but the horizontal symmetry is more important for our purpose). This means for instance that guests 1 and 11 touch exactly the same doors (but not in the same order: while guest 1 touches consecutively the doors from 1 to 12, guest 11 first touches the doors from 11 down to 1, and then finishes with door 12). The same is true for guests 2 and 10, 3 and 9, 4 and 8, and 5 and 7. So this gives us the partition we are looking for: in a hotel with n rooms, guests k and $n-k$ are mates in the sense that if guest k changes the position of a door, then so does guest $n-k$ (but see exceptional values of k below).

The proof of this last claim is a simple generalization of the information contained in the previous table, using some basic notions of modular arithmetic. Consider the activity of guest $n-k$: he touches consecutively door $n-k$, door $2(n-k) \pmod n$, door $3(n-k) \pmod n$, and so on, until he gets to change the position of door n. Look now at the doors touched by guest k, but in reverse order: since he stops as soon as he changes the position of door n, this means that he has just previously touched door $n-k$, and before that, door $(n-2k) \pmod n$, door $(n-3k) \pmod n$, and so on.

To complete the proof of the above conjecture, we just need to look at exceptional cases. And these are of two kinds. First, it is clear that for any n, guest k has no symmetric counterpart when $k = n$: since there is no guest 0, guest $n-k$ simply does not exist in such a case. Secondly, when n is even and $k = n/2$, guest k is the same person as guest $n-k$. So the whole problem boils down to the action of guest n, when n is odd, and to the combined action of guests n and $n/2$, when n is even.

It should be noted that the preceding discussion has given us a "natural" matching between (most of) the guests of Circle Hotel: guest k can be associated with guest $n-k$ (for $k \neq n$ and $n/2$), since, as they touch exactly the same doors, one can be seen as undoing what the other has done. Something similar can also be observed in the case of Long Hotel (example 5), but the matching is not global anymore: it takes place with respect to a given door. Namely, when guest k touches door d, so does guest d/k. The exceptional case now arises when $k = d/k$, i.e., when d is a perfect square.

Let us conclude this example by looking at some questions that arise naturally with respect to Circle Hotel (leaving to the reader the analogous questions for Long Hotel):

(1) from the point of view of a given guest:
  (a) how many times does guest k go round the corridor before stopping?
  (b) how many doors has he touched at the end of his action?
(2) from the point of view of a given door:
  (a) which guests change the position of door d during the process?
  (b) how many are they?

The key to questions 1(a,b) is the fact that the total number of doors in front of which guest k passes during his action in a hotel with n rooms is given by $\operatorname{lcm}(k, n)$. But this number is also equal to the two products $r_k \cdot n$ and $s_k \cdot k$, where r_k is the number of revolutions of guest k around the hotel and s_k is the number of doors he touches during the process. Hence we have expressions for r_k and s_k in terms of k and n.

Question 2(a) can be answered by observing the following fact. If guest k touches door d, this can happen either during his first revolution around the corridor, in which case d is a multiple of k; or during the second revolution, in which case $d+n$ is a multiple of k; or the third, in which case $d+2n$ is a multiple of k; etc. So if guest k touches door d, one can exhibit integers x and y such that $d = xk + yn$. By basic results about gcd's — in particular Bachet-Bézout's relation expressing the gcd of two integers as a linear combination of these —, this amounts to saying that $\gcd(k,n)$ is a divisor of d.

In order to count how many k's are such that $\gcd(k,n) \mid d$ (question 2(b)), we need to evaluate the following sum (in which n and d are fixed parameters and k is the index of summation):

$$\sum_{\substack{1\le k\le n\\ \gcd(k,n)\mid d}} 1\,.$$

In other words, among all the divisors of d, we must concentrate on those which are of the form $\gcd(k,n)$ for a certain $k \le n$. The above sum can then be rewritten as

$$\sum_{x\mid d}\ \sum_{\substack{1\le k\le n\\ \gcd(k,n)=x}} 1\,.$$

It is quite tempting at this stage to rewrite the second $\sum$ so as to exhibit Euler ϕ-function, since $\phi(m)$ counts the integers relatively prime to m and not exceeding it. But in order to do that, we need to record the fact that the index x of the first sum, being equal to $\gcd(k,n)$, must divide n. We thus obtain

$$\sum_{\substack{x\mid d\\ x\mid n}}\ \sum_{\substack{\frac{k}{x}\le\frac{n}{x}\\ \gcd\left(\frac{k}{x},\frac{n}{x}\right)=1}} 1 \;=\; \sum_{\substack{x\mid d\\ x\mid n}} \phi\left(\frac{n}{x}\right).$$

For instance, the number of guests changing the position of door 8 in a circular hotel with 12 rooms is given by

$$\sum_{\substack{x\mid 8\\ x\mid 12}} \phi\left(\frac{12}{x}\right) \;=\; \sum_{x\in\{1,2,4\}} \phi\left(\frac{12}{x}\right) \;=\; \phi(12)+\phi(6)+\phi(3) \;=\; 4+2+2 \;=\; 8\,.$$

The identity of these guests can be read from the above table, column D8.

The problem of Circle Hotel bears analogy with the toy commercialized under the name of "Spirograph". Figure 1 below shows the seven-pointed pattern generated with the Spirograph by rolling the small wheel of 30 teeth inside the big wheel of 105 teeth. This can be seen as a materialization of the behavior of guest 30 in a circular hotel of 105 rooms. Question 1(b) above, for instance, is related to the number of points of the pattern produced.

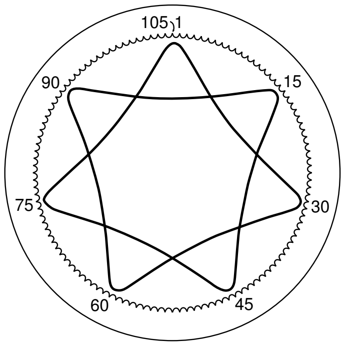
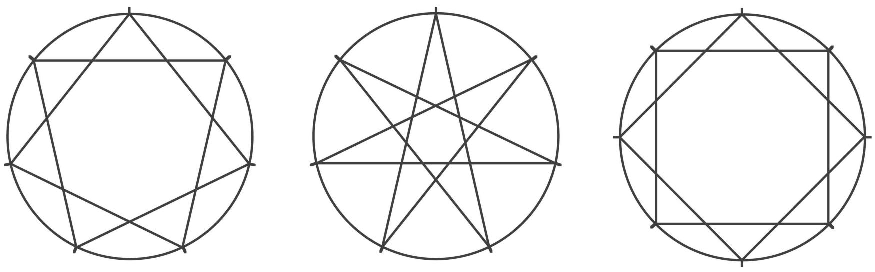


Figure 1

The Circle Hotel problem can also serve as a starting point for the study of *star polygons* $\{n/d\}$: these are (generalized) polygons obtained by connecting with line segments every dth point on a circle where have been marked n equally spaced points (see Figure 2 — the pattern of Figure 1 is thus analogous to the star polygon $\{7/2\}$). Stars polygons constitute an excellent exploratory subject for elementary or secondary school teachers. When n and d are relatively prime, the figure is made of one "orbit" passing through all the n vertices. For n and d having common factors, the resulting figure is compound, consisting of $\gcd(n, d)$ independent orbits. For more details, see Coxeter [8, Section 2.8] and [9, Chapter 6], or Davis and Chinn [10, Chapter 9] — in the latter the expression "Poinsot star" is used to designate such figures.



Star polygon $\{7/2\}$ Star polygon $\{7/3\}$ Compound star polygon $\{8/2\}$

Figure 2

7. The theorem of Pythagoras (P)

The theorem of Pythagoras is probably not a topic fully appropriate for primary school pupils, at least not in its arithmetical disguise. Nonetheless, it is hard to imagine omitting this fundamental theorem with primary school student teachers, as it occupies such a central place in elementary mathematics. One aspect is to make the teachers aware of the link it provides between geometry and arithmetic, with "squares *on* the sides" of a right triangle becoming "squares *of* the sides". But still more basic is the geometrical interpretation of the theorem in terms of sum of areas of squares (see Figure 3 below).

Beyond this interpretation of the theorem stands the question of proving its validity. It is well known that numerous approaches exist to such a proof, one of which is to go back to the original argument of Euclid (*Elements*, Book I, Proposition 47) based on the famous "bride's chair" diagram. While the formal discussion is probably more appropriate for secondary than primary school teachers, the basic geometrical idea underlying Euclid's proof is quite nice, namely that the area of a triangle is not changed when one vertex is moved parallel to the opposite side.

Among the scores of proofs of the Pythagorean theorem accessible in the literature, many rest on a visual argument. The following one is particularly simple, based on the idea of folding Figure 3 along lines AB and AC.

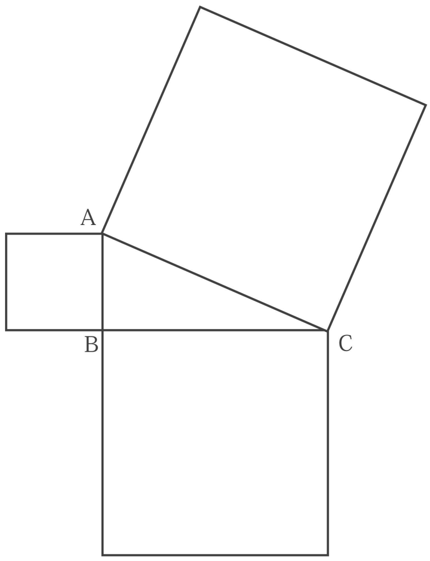


Figure 3

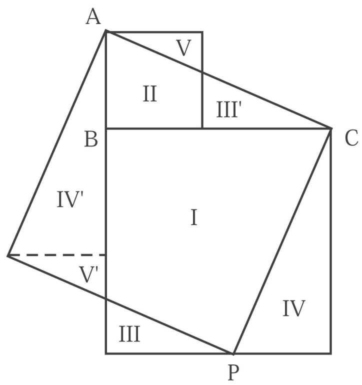


Figure 4

Let us thus reflect the square on AB in this side of the triangle, and similarly for the square on AC. We thus obtain Figure 4, which in itself "is" a proof of the theorem of Pythagoras. What needs to be verified is that reflection of the square on the hypotenuse gives a point P belonging to the square on BC and such that the region IV thus determined is congruent to the original triangle ABC; but this is a direct consequence of inspection of angles and sides of squares. It then follows that regions III and III' are congruent, as are regions IV and IV', and V and V'. Hence the square on AC is equal in area to the sum of the squares on AB and BC respectively.

8. The theorem of Pythagoras, revisited (S)

Here is another (remarkably stripped!) visual proof of the same theorem, presented by Pólya in [25, Section II.5]: just look at Figure 5!

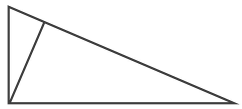


Figure 5

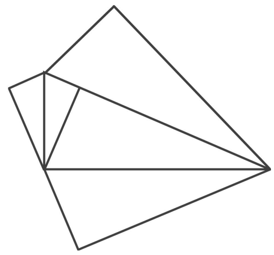


Figure 6

The basic idea behind this visual argument goes back to the following generalization of the theorem of Pythagoras found in Euclid (*Elements*, Book VI, Proposition 31): *In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.* In other words, in spite of the fact that the Pythagorean relation $a^2 + b^2 = c^2$ "speaks of" squares, one does not need to restrict oneself to squares when considering figures on the three sides of the right triangle: any three figures similar one to the others and similarly described can be used — the drawing in the *Elements* VI, 31 shows three similar rectangles. To see why this is true, just notice that transforming a square to a given figure amounts to multiplying its area by a certain factor λ (i.e., λ expresses the ratio of two given areas). Consideration of "similar and similarly described" figures on the three sides of a triangle thus involves the same factor λ. And the Pythagorean relation $a^2 + b^2 = c^2$ is clearly equivalent to $\lambda a^2 + \lambda b^2 = \lambda c^2$, since one can derive the two equations from each other by multiplying or dividing by λ.

So one would like to find "good" figures to be described on the three sides of the given right triangle — good in the sense that the addition-of-area property will become easy to see. Constructing the altitude with respect to the hypotenuse, we see the given right triangle being divided into two smaller ones: this is the clue to the three *similar and similarly described figures* we are looking for (see Figure 6, where these three triangles have been shown unfold outside the basic right triangle). Clearly the large triangle is equal in area to the sum of the other two.

The reader will note that an alternate proof of the theorem of Pythagoras can be obtained from Figure 5 by considering the ratio of the sides of the three similar triangles seen in the figure.

Most probably the reader's favorite proof of the Pythagorean theorem is not among those mentioned above, as the choices are so abundant. My purpose here was not to survey all the "best" proofs, but simply to present a few I find particularly illuminating for teachers.

9. An isoperimetric problem (P)

In an isometric problem, one is asked to find the curve of largest area among curves of a certain type, all being of equal perimeter; for instance, to find the rectangle of a given perimeter P whose area A is maximal. This is a typical problem proposed to beginning calculus students in order to have them apply the optimization techniques developed in such a course (maximize the function $A = x(\frac{P}{2} - x)$). Another approach, in a secondary school setting, is to make use of the so-called Viète's formulas from algebra — i.e., the fact that the roots α and β of the quadratic equation $x^2 + px + q = 0$ are characterized by the equalities $\alpha + \beta = -p$ and $\alpha\beta = q$. Given any rectangle of sides a and b, its half-perimeter $P/2$ and its area A thus satisfy the equation

$$x^2 - \frac{P}{2}x + A = 0\,.$$

But in order for this equation to have real roots, its discriminant must be non-negative, which is the case if, and only if, $P^2 - 16A \geq 0$. Hence the maximal value of A, for a fixed P, is then $P^2/16$, so that the rectangle with maximal area is the square of side $P/4$.

It is possible to solve this problem by a purely geometric elementary approach, making it suitable for primary school student teachers. (This is essentially what is found in Euclid — *Elements*, Book VI, Proposition 27.) Experiments with a few concrete examples lead to the conjecture that the largest rectangle of perimeter P is the square of side $P/4$. In order to compare this square with a typical (non-square) rectangle, one can superpose them, giving Figure 7 below — see Rademacher and Toeplitz [27, Chapter 3]. This figure is typical of "look-see" proofs: it is by itself the complete solution. When passing from a square of side a to a rectangle of same perimeter, one side of the rectangle must be less than a by a certain quantity x, which means that the other side must be greater than a by the same x. The two shaded regions in Figure 7 are thus congruent rectangles. It then follows that the area of the square is larger than that of the rectangle, the difference being the area of the small square of side x. (Note that this difference can be visually identified as follows: it is the area of a square congruent to the "missing" corner in the lower left part of the picture.)

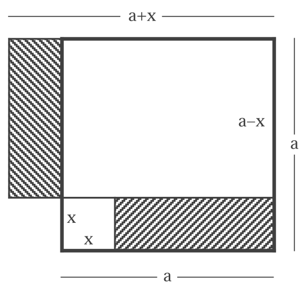


Figure 7

If one wishes to do so, algebraic manipulations could be used, based on Figure 7, to give an alternate proof:

$$\text{area of rectangle} = (a + x)(a - x) = a^2 - x^2 < a^2 = \text{area of square.}$$

10. Other isoperimetric situations (S)

The technique of Figure 7 can readily be adapted to other isoperimetric problems. For instance, suppose a rectangular field is to be determined along a river, three of the sides being made with a fence of a given length (there is no need for a fence along the riverbank). Which should be the lengths of these three sides in order to maximize the area of the field?

Again, experimenting with concrete cases is appropriate to help intuition. This should lead to the following conjecture: the sides of the rectangle must be in the ratio $2:1$, i.e., a fence of length $4k$ should be divided in three parts of lengths k, $2k$ and k, respectively. Figure 8, which is in the spirit of the preceding one, provides a visual proof of this result; we compare, in 8(a), a rectangular field having such proportions with one whose "vertical" sides are smaller than k, and longer in 8(b). (Note again that, in each case, the difference can be visually identified as the area of a rectangle congruent to the "missing" corner in the lower left part of the picture.)

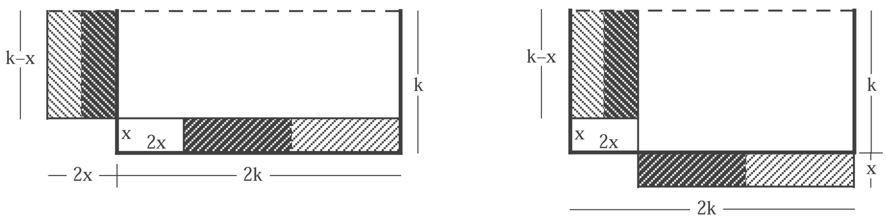


Figure 8

As in the previous case, one can use the variables giving the length of various segments to provide an algebraic proof of the result; for instance, the field enclosed within the fence of ratio $k : 2k : k$ has greater area than the one corresponding to the fence $(k - x) : (2k + 2x) : (k - x)$ (Figure 8(a)), since

$$(2k + 2x)(k - x) = 2k^2 - 2x^2 < 2k^2\,.$$

An alternate way of identifying the proportions determining the maximal area for the field can be obtained by reducing this isoperimetric problem to our original one. Letting a, b and a be the lengths of the three sides formed by the fence, we consider the riverbank as a mirror. We thus get a rectangle of sides $2a$ and b, which is of maximal area when it is square, i.e., for $b = 2a$. Maximality is preserved when "removing" the mirror. (In other words, the rectangular field we are looking for has the shape of half a square.)

Let us conclude this example by recalling a familiar fact, namely that the original isoperimetric problem illustrated in Figure 7 easily leads to a basic result of elementary mathematics which should be familiar to any school teacher.

If we allow, in the algebraic expression $a^2 - x^2 < a^2$ corresponding to Figure 7, the variable x to take the value 0, so to include the square among the possible rectangles, we get

$$(a + x)(a - x) \leq a^2 .$$

Let us introduce a change of variables by calling u and v the sides of an arbitrary rectangle; in other words, we set $u = a+x$ and $v = a-x$, so that $a = (u+v)/2$. Substitution gives

$$uv \leq \left(\frac{u+v}{2}\right)^2 ,$$

that is

$$\sqrt{uv} \leq \frac{u+v}{2} .$$

We have thus recovered the celebrated *arithmetic-geometric mean inequality* stating that the arithmetic mean of two numbers is never smaller that their geometric mean, equality happening only when the two numbers are equal.

Conversely, one could start from this last inequality in order to prove that the square is the solution to our original isoperimetric problem. Squaring the inequality gives $uv \leq \left(\frac{u+v}{2}\right)^2$, which can be read as: the area of a rectangle of sides u and v is never greater that the area of the square of same perimeter. This in turn calls for a direct proof of the arithmetic-geometric mean inequality. Again there are numerous approaches to this result. For instance the visual proof supported by Figure 9 is particularly neat: in a semi-circle of diameter $u+v$, one compares the (perpendicular) radius with the perpendicular raised at the meeting point of the two original lengths, that is, with the mean proportional between the segments of lengths u and v.

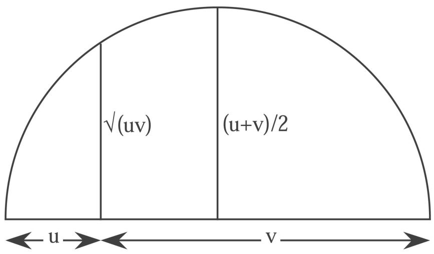


Figure 9

The last few examples have presented situations in which visual reasoning plays a major role. It should be noted that mathematicians have traditionally, in the last centuries at least, been very suspicious of arguments with a strong visual component. But denying the importance of visualization is a terribly sterile standpoint, especially in a pedagogical context. There is a need to educate visual reasoning so to allow recognizing valid and misleading elements in a visual argument, in the same way mathematicians have developed an expertise in judging the validity of linguistic or algebraic reasoning. More on the role of visual reasoning, both in mathematics and in mathematics education, can be found in Dreyfus [11].

11. The six-column sieve (P)

The next example is also an instance of visual reasoning, but in a somewhat different setting. It can be seen as an illustration of the fact that organization of information may be crucial in revealing a pattern. It is again a classical topic of the literature for primary school student teachers — see for instance Bell *et al.* [3, p. 623].

The sieve of Eratosthenes is a well-known device for finding the prime numbers up to a certain limit (say, 99). When asked to accomplish such a task, many people will have the tendency to display the natural numbers in ten columns (with the first row going from 1 to 10, the second from 11 to 20, etc.) — at least if working in base ten. But there is no deep reasons for doing so. Is it possible to identify some more interesting disposition of the numbers, which could make easier sieving the numbers or provide a better insight? It turns out that writing the numbers in six columns is a good way of displaying them.

2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25
26	27	28	29	30	31
32	33	34	35	36	37
38	39	40	41	42	43
44	45	46	47	48	49
50	51	52	53	54	55
56	57	58	59	60	61
62	63	64	65	66	67
68	69	70	71	72	73
74	75	76	77	78	79
80	81	82	83	84	85
86	87	88	89	90	91
92	93	94	95	96	97
98	99				

Search for the multiples of 2 eliminates the first column (except for the top element), the third and the fifth column; search for the multiples of 3 eliminates the second column (except for the top element). So all other primes are to be found in either column four or column six. The following theorem is thus transparent, just from the way the numbers were listed: *Except for* 2 *and* 3*, all primes are of the form* $6k \pm 1$. The proof given here makes this result readily accessible to primary school pupils.

The above table also has the merit that elimination of multiples of other primes is greatly facilitated by the fact each of them is a neighbour of a multiple of 6. For instance, since $5 = 6 - 1$, the multiples of 5 are to be found on slant lines obtained by going down one row and going left one column. And multiples of $13 = 6 \cdot 2 + 1$ are on slant lines obtained by moving down two rows and right one column.

It is clear that the very idea of displaying the natural numbers on six columns comes from the prior knowledge of the fact that primes other than 2 or 3 are of the form $6k \pm 1$. (This result can be easily proved formally by primary school student teachers, for instance by considering a prime p surrounded by its even neighbours $p-1$ and $p+1$ — one of these two must also be a multiple of 3 —, or by recognizing that among the six consecutive numbers $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$ and $6k+5$, four of them are always composite, having 2 and/or 3 as factors.) This situation is particularly instructive in showing student teachers how mathematical knowledge can help them shape their pedagogical approach to given topics.

This example illustrates the fact that a "tool" such as a sieve for prime numbers can be more than a mere device aiming simply at saving labour. It can also be a mind-opener to conceptual knowledge, revealing some hidden connections.

12. The kaleidoscope (P)

Ever since its invention by the Scottish physicist Sir David Brewster in the early 19th century, the kaleidoscope has fascinated people of all ages through the richness and beauty of the pictures created by the interplay of mirrors. Using such an "attention-catcher", teachers can bring their pupils into discovery activities about mathematical topics closely connected to central themes of the school geometry curriculum. The understanding of the mathematical principles underlying the kaleidoscope is a challenge fully appropriate for primary school student teachers. The mastery of such a mathematical "micro-theory" can have a positive impact on their perception of mathematics and their personal relation to it.

The study of the kaleidoscopic phenomenon should go through various phases before a formal description is obtained. After having looked through genuine kaleidoscopes — I always bring my small collection of kaleidoscopes into the classroom when I work this topic —, one should first manipulate real mirrors in a free setting (instead of fixed mirrors in a tube), using small objects or a figure on a sheet of paper as the starting motif placed between the mirrors; it can be observed that while any angle between the mirrors can result in an interesting rose-pattern, it is only for specific angles that "perfectly beautiful and symmetrical forms", as required by Brewster, will be generated. Hence the rules he identified [4] for a "true" kaleidoscope:

- when a symmetric object is place symmetrically between the two mirrors, the angle can be any aliquot part of a circle;
- otherwise, the angle should be an even aliquot part of a circle, i.e., a divisor of $180°$.

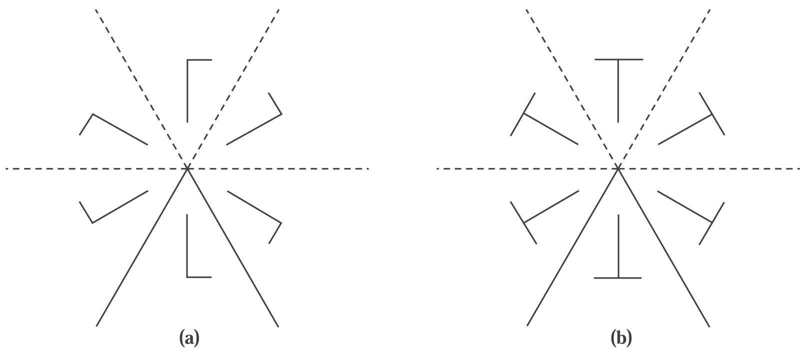


Figure 10

Moving to paper-and-pencil activities, one can abstract the mirrors and their effects to reflections in straight lines; the validity of Brewster's rules then follows from basic properties of transformational geometry (see Hodgson [19] for a guided exploration of this "kaleidoscope geometry"). Figure 10 illustrates the phenomenon in the case of a 60°-degree kaleidoscope. In 10(a), the asymmetric motif "L" has been place between the two mirrors, thus resulting in a 3-fold symmetric rosace. Making this basic motif symmetrical, for instance by extending its "foot" to the left so to get an inverted "T", one obtains the 6-fold symmetric pattern of 10(b). The reader will notice the virtual (dashed) mirrors generated by reflecting the two basic mirrors one in the other and which appear to act like real mirrors. It should also be noted that Figure 10(a) could be obtained with a 120°-degree kaleidoscope, using as the basic motif a combination of the "L" and a mirror-image of it. More generally, any object placed between two mirrors inclined at $180°/n$, for n a divisor of 180°, yields $2n$ images (including the object itself). The symmetry group of the rosace thus generated is the *dihedral group* D_n of order $2n$, consisting of n rotations and n reflections — i.e., the symmetry group of the regular n-gon. See Coxeter [8, Section 2.7] for more information.

A natural extension (discussed by Brewster) is to introduce "polycentral kaleidoscopes" made of three or more mirrors and thus producing clusters of images around several centers spreading in all directions — see Coxeter [9, Section 5.2]. Because of Brewster's rule stipulating that the angle must be a divisor of 180°, only arrangements of three or four mirrors need be considered (in spite of the fact that the regular hexagon tessellates the plane, it does not provide a good combination of mirrors as its angle of 120° will create interference between the images generated). It is noticeable that, in addition to rectangular arrangements of four mirrors, the only other possibilities are to put the three mirrors in the shape of a triangle with angles either 60°–60°–60°, 90°–45°–45° or 90°–60°–30°. This follows from the fact that the only integer solutions of the equation

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1\,,$$

with $a, b, c \geq 2$, are $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$ (or a permutation of these numbers). One immediately recognizes the preceding equation as a direct consequence of a fundamental result of elementary geometry, namely the sum of the interior angles of a triangle. Figure 11, taken from an early article on the kaleidoscope (see [12]), shows a pattern produced with a triangular prismatic kaleidoscope of 90°–60°–30°.

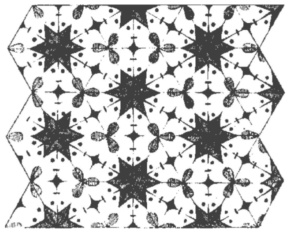


Figure 11

13. Fictitious kaleidoscopes (S)

Another possible level of study of the kaleidoscopic phenomenon results from transferring the constructions to a computer graphics display. For one thing, as is always the case in any programming task, writing a computer program to simulate a kaleidoscope is a very good indication of the understanding one has developed of its principles. But more to the point, the computer environment, in addition to automatizing tedious manipulations and making easily feasible a variety of experiments, also allows situations that are physically impossible. In this last example — the extra in the dozen —, let me indicate some of these situations (a more detailed discussion of the pedagogical environment provided by the computer can be found in Graf and Hodgson [15]).

When two mirrors are placed one near the other, either intersecting or parallel, many images are created all at once — respectively, finitely or infinitely many. With the computer, it is possible to decompose this generation of images in a step-by-step manner, so to see each new image as being obtained from previous ones. In Figure 10(a) above, for instance, reflection of the basic "L" in each of the mirrors gives the two inverted ones in the lower part of the rosace. Reflecting again leads to the two positively oriented sections, in the upper part. Finally, a third level of reflection leads to the "L" up above, again inverted.

The computer also makes easily accessible the study of fictitious kaleidoscopes which have no real material equivalents. Instead of considering the images produced by an axial reflection in a (standard) mirror, let us use a central reflection (reflection in a point instead of a straight line, i.e., half-turn around a given point). Figure 12 below shows patterns generated by prismatic kaleidoscopes built on such reflections — two levels of reflections have been executed in each case, giving nine images from the basic triangle (in bold). In 12(a), the centers of reflection are the three vertices of the basic triangle, while in 12(b) these are the midpoints of the sides. In the former case, we obtain a pattern extending in the plane but leaving blank hexagonal spaces; in the latter, a tessellation is obtained. Moving to quadrilaterals, it can be observed that, with the vertices as centers of reflection, only a few quadrilaterals will give neat patterns: in most cases, the images will be overlapping. On the contrary, when the reflections are taken with respect to the midpoints of the sides, any quadrilateral, even non-convex, will give a tessellation. A proof of the fact that any triangle or quadrilateral can be used as a basic region for a tessellation, when the centers are the midpoints, can be found in Coxeter [8, Section 4.2].

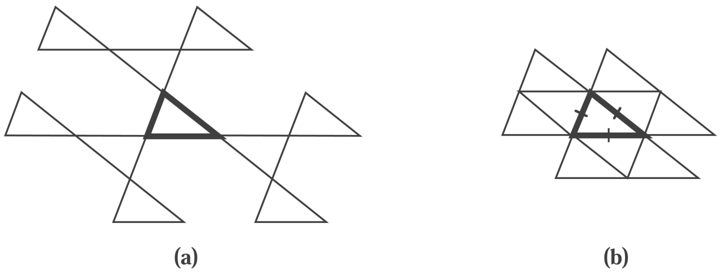


Figure 12

Another possibility offered by the computer is to study transformations for which construction by hands, if feasible, is extremely tedious. A good example of this is the notion of inversion in a circle. While axial and central reflections are standard topics in the school curriculum, circular inversion is not, possibly because of the more complicated mode of construction.

Figure 13 below, taken from Graf and Hodgson [16], was obtained in the following manner. Starting from a circle with its (horizontal) diameter divided into n congruent segments (here, $n = 5$ and 6), $n - 1$ curves connecting the two endpoints of the diameter are drawn inside the circle, each curve being made of two semicircles located each in a different semidisk and whose diameters are obtained by grouping the given segments into two adjacent segments. (The semicircles are drawn in such a way that all those contained in one half of the original disk have in common an endpoint of its diameter.) These curves are then mapped outside the circle through a circular inversion type transformation (here, inversion in the circle, i.e., complex mapping $z \longmapsto 1/\overline{z}$, followed by reflection in the diameter). The resulting pattern can be interpreted as a new type of tessellation, the tiles being bounded by circular arcs and some extending to infinity. In Hodgson and Graf [21], comments can be found on some remarkable geometrical properties enjoyed by such tessellations. A nice feature of the computer is that by making use of the dynamic capacities of software such as Cabri-géomètre, it is possible to explicit the construction in a pointwise manner. Hence, while a point is moving along a given path, one can see the image curve being traced simultaneously, which gives a much better intuition of the process.

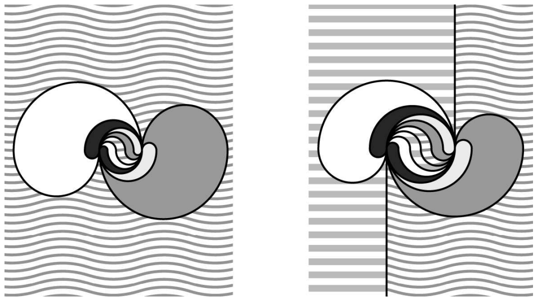


Figure 13

3. Conclusion

An interesting challenge for mathematicians involved in teacher education is to identify topics and problems that can or should be worked with their students. One possible approach is to concentrate on the idea of seeing *elementary mathematics from an advanced standpoint*, of which Felix Klein was a most outstanding promoter — see for instance his lectures [22], given in the early 20th century, in which Klein aimed at providing teachers with a comprehensive view of basic mathematics. Another approach is to identify topics from more advanced mathematics which lend themselves to an elementary discussion; to borrow the title from a remarkable book by Rademacher [26], one now looks at *higher mathematics from an elementary point of view.* The point here is not to trivialize mathematics, but to convey, when possible, the core ideas of a topic in elementary terms, avoiding the use of non-essential sophisticated tools. To a certain extent the above examples concern both these approaches. But clearly my emphasis has been on the former.

This idea initiated by Klein of looking at elementary mathematical topics from an advanced standpoint has proved to be extremely fruitful and can be seen, in many respects, to constitute the core of the mathematical preparation of school teachers: experience shows that it can be truly profitable for students teachers to review from an "adult" perspective basic notions of mathematics they have learned in school. Moreover analyzing elementary mathematics from such an advanced point of view can be crucial for teachers in the development of the conceptual understanding of mathematics essential to the soundness of their pedagogical agenda.

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