Herculean or Sisyphean tasks?

Bernard R. Hodgson (Universit Laval, Qu´ebec, Canada)

Bernard Hodgson

A renewed look at the incompleteness phenomenon in mathematical logic

Hercules or Sisyphus? The hero endowed with exceptional strength and capable of tremendous exploits? Or the ill-fated facing an endless task, condemned to eternally roll up the side of a mountain a rock that hurtles down the slope as soon as the summit is reached? Which of these two characters from Greek mythology spontaneously comes to mind when thinking of some mathematical problems for which it is not known *a priori* whether their solution represents a feasible, though colossal, task, or on the contrary a totally unachievable task?

Let us for instance consider the possibility that an exponent n greater than 2 be such that the equation $x^n + y^n = z^n$ has non-zero integer solutions. A famous proposition, due to Pierre de Fermat (1601–1665), precisely states that such an exponent does not exist. Until the British mathematician Andrew Wiles finally obtained, in 1994,[1] a satisfactory proof of this "last theorem of Fermat", many felt that this might well be the typical case of a Sisyphean situation. Or maybe one should say a "doubly-Sisyphean" situation, since not only did the opponents of Fermat's thesis not succeed in identifying concrete values of n corresponding to solutions, but also all those who over the years had thought that a proof of Fermat's assertion was within reach, had seen their hopes deceived. It is now known, thanks to Wiles, that this problem is rather one whose solution represents a Herculean task: extremely difficult, but nonetheless achievable.

Other situations apparently of a Sisyphean nature, but in fact Herculean, have been brought out in recent years. Quite amazingly links were established between some of these problems and the phenomenon of *incompleteness of mathematical formalism* discovered in the early 30s by the Austrian logician Kurt Gödel (1906–1978). This led to the observation that, contrary to the opinion of a great number of mathematicians or epistemologists, incompleteness does not revolve only around more or less esoteric statements of no concern in the daily work of the mathematician. Recent research has shown that incompleteness is always around and can show up unexpectedly in a host of mathematical contexts. Gone, thus, the cosy indifference of mathematicians towards "metamathematical" considerations! But what exactly does this major change of perspective consist of?

Gödel for all

Metamathematics studies mathematics as it is practised, in particular as regards the nature and the role of formalized reasoning. Taking its roots in the "foundational crisis" triggered off in the early twentieth century by the discovery of inconsistencies in some nooks of the mathematical landscape (*see* the box entitled *Russell Paradox*), the metamathematical enterprise reached its peak in the so-called *Hilbert's Programme*, named after the great German mathematician David Hilbert (1862–1943) who attempted to prove rigorously the non-contradiction of formalized mathematics.

Already with Leibniz (1646–1716) one finds the idea of transposing reasoning to a formal framework through the introduction of appropriate symbolism and deductive rules. And when Hilbert proposes his programme, considerable progress has been accomplished towards this objective, especially since the mid-XIX$^{\text{th}}$ century. But in 1931, *coup de théâtre*, Gödel shakes the scientific community with the publication of his famous *incompleteness theorems*, which reveal strict limitations inherent to mathematical formalism and give, by the same token, a fatal blow to Hilbert's Programme.

The *first* of these theorems establishes the existence, in any formal system satisfying very general conditions — such a system encompasses for instance the theory of *elementary arithmetic* of addition and multiplication, which is really minimal if one is to do any mathematics — of a *statement true but unprovable* according to this system.

As to the *second* incompleteness theorem of Gödel, it provides a specific example of such a statement: namely, the statement expressing the consistency of the formal system itself. And there is no point in trying to remedy such deficiencies by simply adopting these unprovable statements as new basic principles (*axioms*), as new true and unprovable statements will immediately spring up.

[1] The proof announced by Wiles in 1993 contained a flaw which was finally corrected the following year.

Russell Paradox (1902)

Let us designate by C the collection of all sets X satisfying the following property: X is not an element of X. Membership in C thus amounts to making true the defining property; in other words, X is an element of C if, and only if, X is not an element of X. In the particular case where one considers the statement "C is an element of C", one deduces that this assertion is true if, and only if, a second one is true, namely "C is not an element of C". This contradictory situation leads to the conclusion that the defining property of C is unacceptable.
Discovered by the British mathematician and philosopher Bertrand Russell (1872–1970), this paradox shows that "naïve" set theory is inconsistent, so that is it not possible to work with sets without introducing some regulative devices. The goal of the numerous formalizations of set theory developed during the XX$^{\text{th}}$ century is precisely to restrict the rules of formation of sets in such a way to avoid inconsistencies. Metamathematical research, more generally, aims at strengthening the foundations of mathematics and at ensuring the mathematician of a working environment where he does not need to fear encountering a contradiction at any moment.

Reactions to the results of Gödel were quite varied. While some were fascinated by this new awareness of the intrinsic limitations of formalism, the " working " mathematician — namely, the mathematician unconcerned by epistemological matters — considered the incompleteness phenomenon as extraneous to his work. Because either it was the expression of a most reasonable constraint (second theorem) — to be credible, a consistency proof for a formal system can hardly take place inside that system — or else the unprovable statement devised by Gödel in his first theorem, a cunning variation on the theme of the *Liar paradox* (*see* the box *Gödel's unprovable statement*), was perceived as being unsusceptible of being connected to problems " normally " studied in mathematics. This is precisely the aspect about which matters have drastically changed recently. And this is where we meet Hercules and Sisyphus.

Kurt Gödel

Gödel's unprovable statement

It has been known since Antiquity that sentences referring to themselves can lead to odd situations. Such is the case with the assertion " This statement is false ", ascribed to the Greek philosopher Eubulides (IV$^{\text{th}}$ century B.C.). If the assertion is true, then it becomes false, and reciprocally if it is false, then it becomes true. Also based on the notion of *self-reference*, Gödel's statement can be seen as a variant of the Liar paradox. Succinctly it says: " This statement is not provable ". The detailed working-out of Gödel's statement requires a number of rather substantial technical constructions, but it is not too difficult to see why it is both *true* and *unprovable*. If Gödel's statement were provable, it would assert a truth, formalism having precisely been devised so to prove only true sentences. But what does the statement say? That it is unprovable. Hence, if it is provable, it is unprovable, a contradictory situation. One thus concludes that the statement must be unprovable, and consequently true, as it affirms its unprovability. For many, however, Gödel's statement is merely a kind of linguistic game with no link to actual mathematical practice.

When Hercules meets the hydra

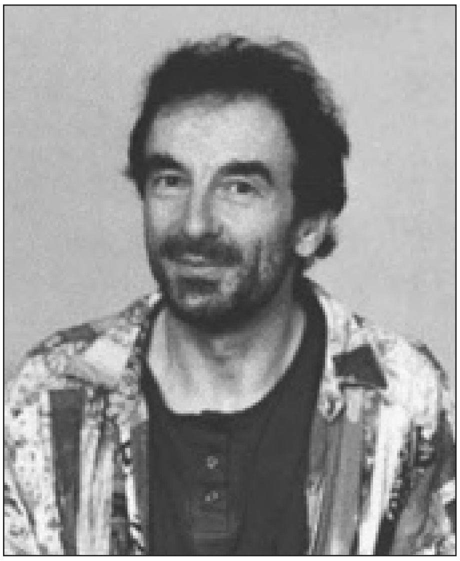
ANTICO (1460 – 1528)
Hercules and the Lernaean Hydra (bronze)

The second of the twelve labours imposed on Hercules was a fight against the Lernæan Hydra, a monster with nine heads, or even more according to some authors, which would grow again as soon as chopped off — in some versions the pruned head was even replaced by two. Legend tells how Hercules finally overcame the Hydra, with the help of his nephew Iolaos who would immediately burn the wound left by a chopped head so to prevent regrowth.

Logicians Laurie Kirby and Jeff Paris have devised a few years ago an even more diabolical hydra to oppose Hercules, as the number of sprouting heads grows as the battle unfolds (*see* the box *The rules of the battle*). In spite of their innocuous appearance the rules defining the battle have a truly dramatic effect as, after only a few steps, Hercules is facing an incredibly dense hydra. Can he nonetheless bring the hydra down? Bluntly said, will he behave as a mythical Hercules or will he be carried, as the pathetic Sisyphus, into an unending chore?

Reuben L. Goodstein (1912 – 1985)

Laurie Kirby



Jeff Paris

The answer is that *Hercules always wins the battle*, whatever the hydra he is confronting or the strategy he uses in cutting off heads (his reputation is thus not overrated!). This most stunning result of Kirby and Paris, based on a remarkable number theorem due to the logician Reuben L. Goodstein [3], fully contradicts the intuition gained from considering a few concrete battles, which appear simply endless — try it and you will see. During his fight against an hydra, Hercules will need to cut an incredibly large, but *finite*, number of heads. Even if the arrangement of heads becomes wider, it actually does not get taller as it forms a tree staying close to the ground, like a bonsai, and with branches connected closer and closer to the root. Patiently, Hercules will end up reducing the hydra to a set of heads directly linked to its root, whence he only needs to sever each head one by one, without any new one being grown.

Sisyphean on its surface, the battle of Hercules against the hydra is thus in fact a genuinely Herculean task: absolutely tremendous in its scope, but still feasible.

Some explosive sequences

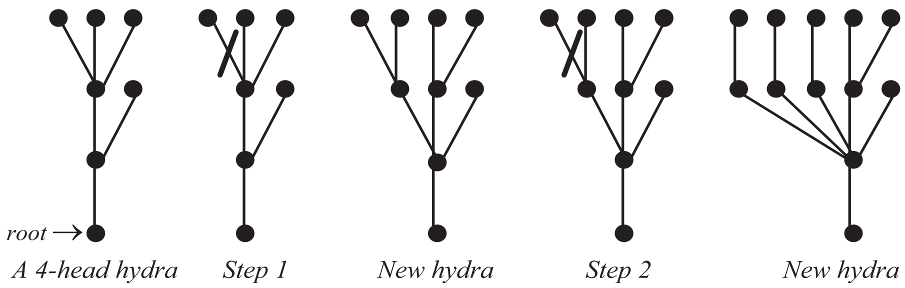
Leaving Hercules to his hydra, let us consider another context of Herculean, but at first glance Sisyphean, tasks. One could think here of Hercules trading the sword for a pencil in order to compute the terms of a numerical sequence.

The number $3 \cdot 2^{402\,653\,211} - 3$ is a gigantic number[2] which made its appearance in the mathematical literature a few years ago. Here is its history.

In 1944, the British logician R. L. Goodstein introduced a process for generating sequences of natural numbers that, against all expectations, inevitably end up in 0. The number $3 \cdot 2^{402\,653\,211} - 3$ is precisely the number of steps needed until the *Goodstein sequence* obtained starting with 4 as the initial value finally reaches 0.

The rules of the battle

A hydra is a tree, that is a configuration of points, each being connected through a unique path to a specific point called the root. A head is a point other than the root located at the end of a path. On the n^{th} step of the battle, Hercules chops off one of the heads (indicated below with an oblique mark), after which the hydra regenerates itself by growing, from the point located two levels below towards the root, n copies of the section containing the pruned head; if the head is directly connected to the root, nothing happens. Hercules wins the battle if the hydra is chopped down to its root.



[2] This number is of the order of $10^{121\,210\,695}$ and comprises more than 121 000 000 digits in its usual decimal representation. If one were to write those digits at the rate of 100 digits per line and 50 lines per page, this would yield a book of more than 24 000 pages!

Weak Goodstein sequences

Let b be a positive integer ≥ 2 (the *base*). Any natural number can then be written (uniquely) in the form

$$m = k_1 \cdot b^{a_1} + k_2 \cdot b^{a_2} + \cdots + k_n \cdot b^{a_n},$$

where the exponents $a_1, a_2, \ldots, a_n$ are strictly decreasing positive integers ($a_1 > a_2 > \cdots > a_n$) and the coefficients $k_1, k_2, \ldots, k_n$ are base-b " digits " — each coefficient is thus a natural number smaller than b. For instance, with $b = 2$ and $m = 266$, one gets

$$\begin{aligned} 266 &= 1 \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= 2^8 + 2^3 + 2^1. \end{aligned}$$

The *weak Goodstein sequence* beginning with 266 is defined as follows. Its first term m_0 is precisely the preceding representation of 266. In order to obtain m_1, the second term of the sequence, one first increases the base by 1, going from 2 to 3, and then subtracts 1 from the result:

$$m_1 = 3^8 + 3^3 + 3^1 - 1 = 3^8 + 3^3 + 2 = 6\,590.$$

One proceeds analogously for computing the other terms of the sequence. After each subtraction of 1, the representation is rewritten, if necessary, as a sum of multiples of powers of the current base — see the computation of m_4 below. Here are the first terms of the weak Goodstein sequence beginning with 266:

$$\begin{aligned} m_0 &= 2^8 + 2^3 + 2^1 \;=\; 266 \\ m_1 &= 3^8 + 3^3 + 2 \;=\; 6\,590 \\ m_2 &= 4^8 + 4^3 + 1 \;=\; 65\,601 \\ m_3 &= 5^8 + 5^3 \;=\; 390\,750 \\ m_4 &= 6^8 + 6^3 - 1 \\ &= 6^8 + 5 \cdot 6^2 + 5 \cdot 6^1 + 5 \;=\; 1\,679\,831 \\ m_5 &= 7^8 + 5 \cdot 7^2 + 5 \cdot 7^1 + 4 \;=\; 5\,765\,085 \end{aligned}$$

A restricted version of Goodstein's process is obtained from the representation of numbers in a specific base. Given a natural number $b \geq 2$, any integer m can be written as a sum of multiples of powers of b. In the case where $b = 10$, this simply yields the usual notation for natural numbers: $266 = 2 \cdot 10^2 + 6 \cdot 10^1 + 6 \cdot 10^0$.

Let us now modify this base-b representation of m by systematically replacing b with $b + 1$, and then subtracting 1 from the result thus obtained. This leads to a new number which can be subjected to a similar process, replacing this time $b+1$ with $b+2$ and then subtracting 1 again (*see* box *Weak Goodstein sequences*).

The successive numbers thus produced appear to grow larger and larger. But this is a misleading observation, as the phenomenon is temporary: any such sequence, if pursued long enough, will unavoidably get to 0. This is without any doubt a most astonishing result, and it reflects the fact that in spite of the increase in the base, the subtraction of 1 gradually " eats away " one after the other, all the terms appearing in the successive expressions.[3]

The process studied by Goodstein is in fact more general and involves much more spectacular numbers. It rests on the notion of the *complete representation* in base b of a natural number: as in the preceding case, one writes this number as a sum of multiples of powers of b, but then does the same with the exponents occurring in this representation, as well as with the exponents of these exponents, etc., until the whole representation stabilises. For example, the complete base-2 representation of 266 ($= 2^8 + 2^3 + 2^1$) is $2^{2^{2+1}} + 2^{2+1} + 2^1$.

Goodstein's process consists in replacing once again by $b + 1$ all instances of b in the complete base-b representation of an integer, and then subtracting 1. The growth which can be observed in the first terms of the resulting sequences (*see* the box *Goodstein sequences*) is simply phenomenal when compared to weak sequences, the terms rapidly becoming of a truly breathtaking size!

Still here, despite the fantastic explosion that can be observed, a " Hercules " computing the successive terms of such a sequence cannot avoid eventually obtaining 0, a fact which, it must recognised, is totally counterintuitive. Goodstein [3] has indeed proved that any Goodstein sequence reaches 0 — but this will in general take a very long time! For instance, beginning with $m = 4$, the rank of the term m_r where the sequence finally gets to 0 is precisely the number $r = 3 \cdot 2^{402\,653\,211} - 3$ mentioned above.[4] And the effect is still more fantastic if one takes a larger integer as the source of the sequence .

This is yet another instance of a task apparently of a Sisyphean nature, but which is in fact Herculean: computing the successive terms of a Goodstein sequence seems to be a never-ending job, and it will indeed be extremely long, but it is inevitably finite as one always gets to 0… as long as one is patient enough!

[3] Note that even if some computations make the terms sometimes more numerous — see the passage from m_3 to m_4 in the box *Weak Goodstein sequences* —, their exponents get smaller. This situation is a reminder of the heads of the hydras, which grow in number but inexorably get closer to the ground.

[4] This result is not too difficult to establish and is the topic of a guided exercise in [4].

Goodstein sequences

Given the natural number m, the *Goodstein sequence* beginning with m is the sequence of integers $m_0, m_1, m_2, \ldots$ defined as follows:

- m_0 is the complete base-2 representation of m;
- m_1 is obtained from m_0 by replacing all 2's by 3, and then subtracting 1;
- m_2 is obtained from m_1 by replacing all 3's by 4, and then subtracting 1;

etc., the process terminating if a term takes the value 0.
The first terms of the Goodstein sequence beginning with $266 = 2^{2^{2+1}} + 2^{2+1} + 2^1$ are

$$\begin{aligned}
m_0 &= 2^{2^{2+1}} + 2^{2+1} + 2^1 \ = \ 266 \\
m_1 &= 3^{3^{3+1}} + 3^{3+1} + 3^1 - 1 \\
&= 3^{3^{3+1}} + 3^{3+1} + 2 \\
&= 443\,426\,488\,243\,037\,769\,948\,249\,630\,619\,149\,892\,886 \ \approx \ 10^{38}, \\
m_2 &= 4^{4^{4+1}} + 4^{4+1} + 1 \ \approx \ 10^{616}, \\
m_3 &= 5^{5^{5+1}} + 5^{5+1} \ \approx \ 10^{10\,921}, \\
m_4 &= 6^{6^{6+1}} + 6^{6+1} - 1 \\
&= 6^{6^{6+1}} + 5\cdot 6^6 + 5\cdot 6^5 + 5\cdot 6^4 + 5\cdot 6^3 + 5\cdot 6^2 + 5\cdot 6^1 + 5 \\
&\approx 10^{217\,832}, \\
m_5 &= 7^{7^{7+1}} + 5\cdot 7^7 + 5\cdot 7^5 + 5\cdot 7^4 + 5\cdot 7^3 + 5\cdot 7^2 + 5\cdot 7^1 + 4 \\
&\approx 10^{4\,871\,822}.
\end{aligned}$$

It is not easy to provide a simple explanation for the strange behaviour of Goodstein sequences. It should be stressed however that while the exponential part of Goodstein's process seemingly gives rise to a real numerical outburst, this is nonetheless a limited phenomenon, as the boundless growth is only apparent. As in the weak case, subtraction of the unit will eventually gobble up the gigantic numbers resulting from the successive changes in the base — but this may happen only at the end of a very long journey, because of the staggering values met along the road. Let us observe for instance the impact of this “ -1 ” on the Goodstein sequence beginning with 3, an exceptionally short one:

$$\begin{aligned}
m_0 &= 2^1 + 1\cdot 2^0 \\
m_1 &= 3^1 + 1\cdot 3^0 - 1 \ = \ 3 \ = \ 1\cdot 3^1 \\
m_2 &= 1\cdot 4^1 - 1 \ = \ 3 \ = \ 3\cdot 4^0 \\
m_3 &= 3\cdot 5^0 - 1 \ = \ 2 \ = \ 2\cdot 5^0 \\
m_4 &= 2\cdot 6^0 - 1 \ = \ 1 \ = \ 1\cdot 6^0 \\
m_5 &= 1\cdot 7^0 - 1 \ = \ 0.
\end{aligned}$$

As the term $m_2 = 3\cdot 4^0 = 3$ is, so to say, independent of the base 4, the next terms are not affected by the following changes in the base, so that the sequence merely decreases to 0. The very same phenomenon will take place for any Goodstein sequence, but with terms of a much higher rank: it is for instance remarkable how much longer the Goodstein sequence beginning with 4 is, when compared to that whose origin is 3.

The behaviour of Goodstein sequences is reminiscent of the battles of Hercules with the hydra. In both cases one faces a situation which appears to explode uncontrollably. The fact is, however, that while both contexts are extraordinarily complex, they relate to processes which are finite as they do terminate after a certain time — extremely long! Apparently Sisyphean, the corresponding tasks are thus genuinely Herculean.

But there is more to it: the Herculean-Sisyphean dichotomy, whether in the context of the hydra or of Goodstein, takes on a new flavour when approached from another perspective bringing the incompleteness phenomenon to the fore.

And what about incompleteness?

The number of heads chopped off by Hercules during a battle is absolutely enormous: finite, but exceeding without any doubt the wildest imagination. For instance, even if Hercules would eliminate heads at the furious pace of one head severed every second, a fight against a modest hydra with only a few heads at the outset could last so long that the number of seconds since the Big Bang would in comparison appear trivial! And it is precisely in this aspect of the situation that incompleteness lies.

Taking, as suggested above, elementary arithmetic as a framework, one has the following double phenomenon. It is possible, on the one hand, to simulate in that context Hercules' battles via “ number crunching ”: this is a technical, but not too difficult fact (one associates to each hydra a suitable numeric code). It is however *impossible* to prove in this setting that Hercules always wins: a rigorous proof of this invincibility requires richer contexts, such as the theory of *transfinite ordinal numbers*. It can in fact be proved that there is a limit to the growth rate of functions which can be dealt with in elementary arithmetic, and the function giving the length of the battles precisely exceeds this limit: it grows much too quickly! In a similar way, the function expressing the length of Goodstein sequences, that is the number of terms needed to reach 0, always takes finite values, but its growth goes well beyond the framework of elementary arithmetic.[5]

While our “working” mathematician tends to judge as artificial the statement devised by Gödel, he perceives the Hercules *vs* hydra fights or the computations of Goodstein sequences as *bona fide* mathematical problems — belonging to the branch of mathematics called *combinatorics*. This mathematician knows how to prove with appropriate tools that the battles always end in Hercules’ victory and that Goodstein sequences all converge to 0, but he cannot do that in elementary arithmetic, which is nevertheless the natural formal setting for combinatorics. He is thus facing a proposition which is *true* but *unprovable* in the framework provided by the theory to which it belongs. While the battles against hydras or the computations *à la* Goodstein are in fact Herculean, they will appear Sisyphean for anyone taking elementary arithmetic as the “system of reference”. This state of affairs is thus tinged with a kind of relativity: the task appears endless for an observer located “inside” elementary arithmetic, but it is in fact finite — though colossal —, which can be noticed by looking at things from “outside” arithmetic, in a more powerful formal setting.

Fermat’s last theorem belongs as well, by content, to elementary arithmetic; but the proof given by Wiles rests on high-level mathematical tools, quite beyond that theoretical framework. Is there hope that Fermat’s assertion, while Herculean, could be proved one day to be regarded as Sisyphean when considered from the viewpoint of elementary arithmetic? Such a result would be most revealing, as it would thus provide a kind of measure of the intrinsic difficulty of this theorem. While this possibility is not excluded, research in mathematical logic is however not quite yet at that stage.

Bibiography

1. Cichon, E. Adam. A Short Proof of Two Recently Discovered Independence Results Using Recursion Theoretic Methods, *Proceedings of the American Mathematical Society*, **87**, 704-706, (1983)

2. Gardner, Martin. *The Last Recreations*, New York, Springer-Verlag, 27-43, (1998) See Chapter 2, Bulgarian Solitaire and Other Seemingly Endless Tasks (first published in *Scientific American*, 12-21, August 1983).

3. Goodstein, Reuben L. On the Restricted Ordinal Theorem, *Journal of Symbolic Logic*, **9**, 33-41, (1944)

4. Henle, James M. *An Outline of Set Theory*, New York, Springer-Verlag, (1986)

5. Hofstadter, Douglas R. *Gödel, Escher, Bach: An Eternal Golden Braid*, New York, Basic Books, (1979)

6. Kent, Clement F. and Hodgson, Bernard R. Extensions of Arithmetic for Proving Termination of Computations, *Journal of Symbolic Logic*, **54**, 779-794, (1989)

7. Kirby, Laurie and Paris, Jeff. Accessible Independence Results for Peano Arithmetic, *Bulletin of the London Mathematical Society*, **14**, 285-293, (1982)

To go further

Hofstader [5] presents a popular exposition of Gödel’s results and also the development of an analogy with the music of Johann Sebastian Bach and the drawings of the Dutch artist Maurits C. Escher (1898–1972). Our mythological analogies take their origin in the technical paper [7], in which Kirby and Paris presented their results, as well as in Gardner’s popular account [2]. The distinction between weak and general Goodstein sequences can be found in Cichon [1]. Paper [6] explores the possibility of using properties of Goodstein sequences to define formal frameworks where one can prove the finiteness of some algorithmic processes studied in theoretical computer science.

Acknowledgement. This paper has been prepared for the “RPA-competition” organised by the Committee for Raising Public Awareness of Mathematics of the European Mathematical Society and is based on material which has appeared, in French, in the book “Mathématiques d’hier et d’aujourd’hui”, Richard Pallascio and Gilbert Labelle, eds., Modulo Éditeur, 2000. Reproduced with permission of the copyright owner.

Bernard R. Hodgson [bhodgson@mat.ulaval.ca] did his PhD in mathematical logic at the Université de Montréal. He has been since 1975 at Université Laval (Québec, Canada), Département de mathématiques et de statistique, where he currently is Professeur titulaire. *Besides his research in mathematical logic and theoretical computer science, he is highly involved in mathematics education, in particular as regards the mathematical preparation of primary and secondary school teachers and the use of the history of mathematics in mathematics education. He is the current Secretary-General of the International Commission on Mathematical Instruction (ICMI).*

[5] On the contrary, elementary arithmetic allows to prove that weak Goodstein sequences are always finite. The underlying process is thus somewhat complex, but in practice much less than in the case of general Goodstein sequences.