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THE MATHEMATICAL EDUCATION OF SCHOOL TEACHERS: ROLE AND RESPONSIBILITIES OF UNIVERSITY MATHEMATICIANS

1. INTRODUCTION

The preparation of mathematics teachers for the primary and secondary school is a multifaceted task. It spans various periods of the teachers' life, encompassing their experiences first as pupils, then as undergraduate students and finally as professionals learning from their own action and from in-service activities. Although relatively short, the formal part of this preparation is definitely crucial. Besides the need to include actual teaching practice as early as possible (i.e., working with pupils in real classrooms), three components of this preparation can be identified, which reflect the belonging and interests of the university educators responsible for the formal education of teachers: mathematics itself, didactics of mathematics and psychology of learning. The focus of this paper is on the first of these components.

Mathematicians have a major and unique role to play in the education of teachers — they are neither the sole nor the main contributors to this complex process, but their participation is essential. Maybe this will be seen as a truism, at least in connection with the preparation of secondary school mathematics teacher. But I wish nonetheless to present here some comments about the context in which this role can and should be played. I also want to support the view that mathematicians should take part in the education of primary school teachers. I see such an involvement as important because of the perspective on mathematics itself mathematicians can bring to student teachers. Moreover, I believe this involvement can be a source of gratifying and stimulating mathematical moments for the mathematicians themselves.

In the final part of this paper, I will briefly suggest a few examples of mathematical topics which, from my experience, nicely illustrates the richness of the mathematical content pertaining to student teachers, both of the primary and secondary level. But first I want to examine some aspects of the role and responsibilities of mathematicians in the preparation of schoolteachers, in particular from an historical perspective.

2. MATHEMATICIANS AND TEACHER EDUCATION

Discussing teacher education from an international perspective is always a challenge. Differences are so great from one country to another, due to economical,

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social and cultural factors as well as local traditions, that generalisations are almost impossible. Even within a single country the models are sometimes extremely varied.

In this paper, I want to concentrate on the problem of pre-service mathematical preparation of primary and secondary school teachers, which is typically done in colleges, universities or institutes for teacher education. The mathematicians who are at the centre of my discussion may thus belong to various units within a university. While some of them are attached to educational divisions, either directly or through cross-appointments, the majority of the mathematicians I have in mind are members of departments of mathematics. A key issue is then to what extent their department is really supportive of their involvement, actual or potential, in teacher education.

It does not make much sense to hope for every individual in every department of mathematics to develop interest or expertise in teacher education. For instance, it would be unreasonable to expect a young Ph.D. in pure mathematics with a high proficiency in research, who has been hired in a top-level department world-renowned for research, to devote much energy to the education of teachers. Even in more typical departments, the duties to be fulfilled year after year are in most cases extremely varied and each faculty member tends to develop some speciality. The question, I believe, is whether the task of teaching teachers is regarded as *bona fide* work in mathematics, on the same level, say, as teaching to engineers or to students specialising in mathematics. Looking at various indicators (for instance, peer recognition, promotions, grants, etc.), it is quite tempting to conclude that teaching mathematics to prospective teachers, and especially to those of the primary school, is regarded by many as low-level work in comparison, for example, with teaching to undergraduate mathematics students or still more to graduate students — not to speak of mathematical research *per se*. However, such a view denies the fact that the quality of the students in university classrooms is conditioned to a large extent by the quality of the teachers they have had in school, even at the primary level.

There is, within the mathematics community, a very long tradition of mathematicians getting seriously involved not only in general pedagogical issues, but also more particularly in the problems related to the preparation of schoolteachers. I will now recall briefly a few historical facts related to this. I will refrain in the present text from commenting on the mathematicians' role in education from an epistemological perspective; such a view, presented in connection with the nature of mathematics itself and its contribution to society, can be found in de Guzmán (1993). See also the influential paper of Thurston (1990).

2.1 Some historical milestones

It might be an interesting task to trace back to past centuries the influence exerted on educational matters by mathematicians, either directly or indirectly. But I will restrict my comments mainly to a few events that took place in the last century.

The history of the International Commission on Mathematical Instruction (ICMI) is a rich source of information on the implication of mathematicians in education (see Howson, 1984, for a description of the origins, history, work and aims of ICMI). In fact, one could even see ICMI as having been formed on the very assumption that university mathematicians should have an influence on school mathematics — at least at the secondary level. This is suggested by the following resolution adopted during the fourth International Congress of Mathematicians, held in Rome in 1908, and which marks the inception of ICMI:

> The Congress, recognizing the importance of a comparative study on the methods and plans of teaching mathematics at secondary schools, charges Professors F. Klein, G. Greenhill, and Henri Fehr to constitute an International Commission to study these questions and to present a report to the next Congress. (Lehto, 1998, p. 13)

This resolution was submitted on the initiative of David Eugene Smith, who had suggested the idea of such a commission three years earlier in the international journal *L'Enseignement Mathématique* (see Smith, 1905), in response to a survey proposed by the editors of the journal, Charles-A. Laisant and Henri Fehr, on reforms needed in mathematics education.

Since the establishment of ICMI, many distinguished mathematicians have played a major role in it. The list of officers of the Commission is quite eloquent in this respect — to give just a few names (see Howson, 1984): Felix Klein, Jacques Hadamard, Marshall H. Stone, André Lichnerowicz, Hans Freudenthal, Hassler Whitney, Peter Hilton, Jean-Pierre Kahane; or the current ICMI President, Hyman Bass.

The birth of ICMI took place at a time when many events with an educational flavour were happening in various countries. For instance, the journal *L'Enseignement Mathématique* had been founded in Switzerland a few years earlier. In France, an extensive reform of national curricula, concerning especially the teaching of geometry, had been set forth in 1905 by the government. Many mathematicians, such as Émile Borel, were then involved in discussions about the teaching of elementary mathematics (see Borel, 1914). But the most striking example is surely that of Felix Klein in Germany.

Already, during the last decade of the nineteenth century, Klein was very active in making accessible to secondary school teachers some of the most recent mathematical developments of his time. For example, he presented in his 1894 summer course for teachers (*Vorträge über ausgewählte Fragen der Elementargeometrie*, published in 1895 and soon to be translated into French, Italian, English and Russian), a proof of the impossibility of the three Greek geometrical constructions, Cantor's proof of the existence of transcendental numbers, as well as proofs of the transcendence of e and π. These lectures, he wrote, were due to his desire "to bring the study of mathematics in the university into closer touch with the needs of the secondary schools" (Klein, 1897, p. 1) and they are still considered, a century later, as a masterpiece of exposition and an excellent entrance to modern science (see Kahane, 1997, p. ii).

Some ten years later, in his lectures entitled *Elementarmathematik vom höheren Standpunkte aus*, Klein intended to provide teachers with a comprehensive view of basic mathematics (Arithmetic, Algebra, Analysis, Geometry) of such a range, he wrote in the Preface, "as I should wish every teacher in a higher school to have" (Klein, 1939, p. v). His concern was not "with the different ways in which the problem of instruction can be presented to the mathematician", but rather

> with developments in the subject matter of instruction. I shall endeavor to put before the teacher, as well as the maturing student, from the view-point of modern science, but in a manner as simple, stimulating, and convincing as possible, both the content and the foundations of the topics of instruction, with due regard for the current methods of teaching. (Klein, 1932, p. iii)

He concluded his introductory remarks with the wish that his exposition of elementary mathematics

> may prove useful by inducing many of the teachers of our higher schools to renewed use of independent thought in determining the best way of presenting the material of instruction. This book is designed solely as such a mental spur, not as a detailed handbook. The preparation of the latter I leave to those actively engaged in the schools. (Klein, 1932, p. iv)

It was Klein's hope that this "renewed use of independent thought" may help teachers overcome the "double discontinuity" which they meet when going from secondary school to university, and then back to school as a teacher (see Klein, 1932, p. 1). As commented by Furinghetti (2000), teachers, as a consequence of this double discontinuity,

> tend to reproduce in teaching what was taught them in secondary school. This is one of the causes of the conservative style characterizing school systems around the world. It is because of this conservative style that changes in the curricula or in the method of teaching are so difficult to make.

The idea initiated by Klein of looking at elementary mathematical topics from an advanced standpoint has proved to be extremely fruitful and can be seen, in many respects, to constitute the core of the mathematical preparation of schoolteachers.

The list of prestigious mathematicians significantly involved in matters related to education could be extended much further. Leonhard Euler, for instance, while not a teacher in the usual sense of that word today, surely possessed an acute sense of the importance of expository quality, as can be judged from such gems as his *Letters to a Princess of Germany*, his *Elements of Algebra*, or his *Foundations of Differential Calculus*, to name three of his masterly written elementary works. As an example of more recent times, Henri Poincaré, one of the most distinguished mathematicians at the turn of the century, enjoyed confronting pedagogical issues, as is testified to by some of his papers published in *L'Enseignement Mathématique* during its first decade and devoted to topics such as the links between differential notation and teaching (vol. 1, 1899), logic and intuition in mathematics and in teaching (vols. 1, 1899 and 10, 1908), or the role of definitions in mathematics (vol. 6, 1904). (This perception of pedagogically inclined distinguished mathematicians could be challenged with statements such as “I have a real aversion to teaching” (Gauss, 1802) or even “I hate ‘teaching’” (Hardy, 1967, p. 149). But this in turn could be contrasted with a famous assertion from Poisson (see Hauchecorne and Suratteau, 1996, p. 287) claiming that life is good for only two things: discovering mathematics and teaching mathematics.)

I would now like to turn to a major mathematician, George Pólya, who exerted an exceptional influence in the educational realm during the last half-century (the first edition of the celebrated ‘How to Solve It’ appeared in 1945).

2.2 Pólya on the education of teachers

The impact of Pólya’s works in general mathematics education is remarkable and has been presented in various places (see, for instance, the ICMI Presidential Address of Kahane (1988), which also includes comments on his mathematical works). His views on the teaching of mathematics, and on the consequential question of the mathematical preparation of mathematics teachers, have been examined in depth by Kilpatrick (1987). I would like to point here to just a few aspects of these.

The basis of Pólya’s ideas about the education of mathematics teachers is probably to be found under the item *Rules of teaching* of his ‘Short dictionary of heuristic’ (Pólya, 1957, p. 173):

> The first rule of teaching is to know what you are supposed to teach. The second rule of teaching is to know a little more than what you are supposed to teach. … [I]t should not be forgotten that a teacher of mathematics should know some mathematics, and that a teacher wishing to impart the right attitude of mind toward problems to his students should have acquired that attitude himself.

The same ideas come back in his ‘Ten commandments for teachers’ (see Pólya, 1959, and Pólya, 1981, p. 116) under items 1, 2 and 5: (1) Be interested in your subject; (2) Know your subject; and (5) Give your students not only information, but *know-how*, attitudes of the mind, the habit of methodical work.

Knowledge, in Pólya’s view, consists partly of *information* and partly of *know-how*, and teachers need to impart both. Mathematical know-how has been described by Pólya as “the ability to solve problems, to construct demonstrations, and to examine critically solutions and demonstrations”. And he imposes on these two components of knowledge a very sharp hierarchy: “know-how is much more important than the mere possession of information”. Hence his somewhat provocative conclusion: “it may be more important in the mathematics class how you teach than what you teach”. (Quotations from Pólya, 1959, p. 528, and Pólya, 1981, p. 118.)

Pólya cannot help observe that secondary school mathematics teachers are very badly equipped when they begin their teaching career. Having themselves left “the high school, more often than not, with no knowledge or with a wobbly knowledge of high school mathematics" (Pólya, 1959, p. 531), the question arises: where and when should they acquire the knowledge of the mathematics necessary for their teaching? And more precisely: where and when should they not only learn basic facts or improve their skills in secondary school mathematics, but, more to the point, develop know-how, i.e., their ability to reason and think creatively?

As commented by Kilpatrick (1987, p. 87), Pólya's appreciation of the teacher preparation offered by colleges or universities is that it does very little to improve this situation: "In Pólya's view, departments of mathematics stress abstruse information at the expense of developing mathematical know-how, whereas schools of education dwell too heavily on content-free teaching methods."

The remedy that Pólya advocated was to give prospective teachers an opportunity to pursue mathematical investigations of their own at a level appropriate to their interest and expertise: this was the idea behind his seminar in problem-solving for teachers, where the requisite knowledge was at the high school level and the difficulty of the problems to solve, just a little above high school level (Pólya, 1959, p. 532). Pólya has presented in great detail the philosophy underlying this seminar and the types of problems he worked with teachers (see Pólya, 1981).

The creativity involved in such an approach can help the teachers develop a dynamic relation to mathematics and eventually bring some original contribution to it. Cassidy and Hodgson (1982) have suggested that finding new solutions to given problems, and in particular 'elementary' solutions requiring a minimal level of prior knowledge or even of mathematical maturity, could be seen as a type of research activity fully appropriate for teachers. A parallel can be made here with the merit, in traditional mathematics, of finding a new proof of an already established theorem. On this matter, Wittgenstein has maintained the following:

> It might be said: ' — that every proof, even of a proposition which has already been proved, is a contribution to mathematics'. But why is it a contribution if its only point was to prove the proposition? Well, one can say: 'the new proof shews (or makes) a new connexion'. (Wittgenstein, 1978, p. 191)

More on Pólya's views about the mathematical preparation of secondary school mathematics teachers can be found in Chapter 14 of Pólya (1981) entitled 'On learning, teaching, and learning teaching'. (Although many of Pólya's ideas concerning the education of teachers could obviously be extended to primary school teachers, Pólya did not, as far as I know, write explicitly about them.)

2.3 Towards a conceptual understanding of mathematics

It is quite common to find among the general public a distorted image of mathematics, a widespread view reducing it to rote calculations. Prospective teachers may also share such biases to a certain extent, even those at the secondary school level. The formal preparation offered to teachers at the university should thus be seen as a unique occasion to help them develop a better appreciation of what mathematics really is. While furthering the development of know-how advocated by Pólya, this preparation should put them in contact with and see in action some of the 'deep ideas' which form the heart of mathematics (see Steen, 1990): *structures* (numbers, shapes, algorithms, functions, …) ; *attributes* (linear, periodic, symmetric, random, …) ; *actions* (experiment, model, classify, prove, …) ; *abstractions* (symbols, infinity, similarity, recursion, …) ; *attitudes* (wonder, beauty, …) ; *behaviours* (motion, convergence, stability, chaos, …) ; *dichotomies* (discrete–continuous, finite–infinite, …).

Reports have been published recently, which provide a survey of several researches confirming the importance of teachers' knowledge of mathematical content (see Brown and Borko, 1992, Fennema and Franke, 1992, Grossman, Wilson and Shulman, 1989; Swafford, 1995). For instance, a contrast is made in Grossman et al. (1989) between one teacher with a weak knowledge, who drilled pupils in algorithms to be memorised and applied to predictable problem sets, and another one with a strong mathematical background, who emphasised the 'whys' of mathematics and led pupils to think through problems. Moreover, not only do teachers with a shallow knowledge constitute inadequate models for their pupils of the attitude of mind to develop, but also, as they need to fill holes in their content knowledge, they are bound to have less time to devote to properly pedagogical issues:

> Without adequate content knowledge, student teachers spend much of their limited planning time learning content, rather than planning how to present the content to facilitate the [pupils'] understanding. Student teachers with strong content preparation are more likely to be flexible in their teaching and responsive to [pupils'] needs, and to provide conceptual explanations, instead of purely procedural ones. They also tend to place greater emphasis on the organization and connectedness of knowledge within the discipline and less on the provision of specific information. Student teachers without adequate content knowledge are likely to lack confidence in their ability to teach well. (Brown and Borko, 1992, p. 220)

Teachers with weak competencies in mathematics may be unable to reach the level of autonomy enabling them to appraise critically the adequacy and accuracy of textbooks or to distinguish between more or less legitimate claims. They lack the expertise to identify 'good ideas', as they can only tell the difference between a familiar and an unfamiliar situation. As commented by Moise (1984), "[w]hen unfamiliar insights are ignored as if they were worthless, or actually rejected as wrong, the effect is to discourage students, or corrupt their mentality, or both."

The expression *conceptual understanding* is often used to describe the type of knowledge of mathematics a teacher needs to develop in order to fully play the role of 'facilitator' between pupils and mathematical knowledge. Going much beyond mere factual information, conceptual understanding of mathematics stresses organising principles and central concepts. It allows teachers to perceive mathematics not as a set of facts to be memorised, but as a co-ordinated system of ideas. It makes explicit the paradigms proper to mathematics and shows proofs and reasoning, in various forms, playing a crucial role in certifying facts. It gives a thread linking key mathematical ideas. Such a notion of conceptual understanding is at the core of the famous study of Ma (1999), comparing teachers from China and the USA with respect to a "profound understanding of fundamental mathematics".

I would like to illustrate this point with an elementary example. What most adults remember about the notion of area they learned at school is a list of a few formulas: formula for the area of a square or a rectangle with respect to the sides, for the area of a triangle with respect to a side and the corresponding altitude, etc. The *Tangram* is a familiar geometrical puzzle made of seven geometrical shapes which provides a very nice setting for allowing prospective primary school teachers to better understand the concept of area. Many of them are at a loss when asked, for instance, to find the area of the large triangle in terms of the parallelogram (these are two of the seven pieces): where are the formulas? This indicates a confusion between the concept of *measuring* an area (i.e., comparing a certain surface with a given unit-surface, eventually via intermediate units), and that of *calculating* an area (i.e., replacing the measurement of an area by measurements of lengths followed by arithmetical manipulations on these measures, according to some appropriate formula). A teacher must be aware of the fact that, while reducing an area problem to an arithmetical calculation is often a useful device, it does not reflect the quintessence of measurement of area *per se*.

As can be expected, the more teachers are comfortable with the concepts of mathematics in a deep way, the better they are equipped to work these concepts with their pupils: "The evidence is beginning to accumulate to support the idea that when a teacher has a conceptual understanding of mathematics, it influences classroom instruction in a positive way." (See Fennema and Franke, 1992, p. 151.) However, research also suggests that prospective mathematics teachers are often lack adequate content knowledge, especially those of the primary level who in addition frequently show high levels of 'math anxiety' — see Brown and Borko (1992, p. 220) and Fennema and Franke (1992, p. 148). This leads us to the difficult question of identifying the type of mathematical experiences prospective teachers should meet during their formal education in universities or colleges.

2.4 Programmes for prospective teachers

Programmes for prospective teachers of mathematics must address the many dimensions of the teachers' task, preparing them for the decision-making process in which they will be involved daily in their classrooms. Some of the teachers' major roles, as identified in the NCTM report 'Professional Standards for Teaching Mathematics' (see NCTM, 1991, p. 5) are: setting goals and selecting or creating mathematical tasks to help students achieve these goals; stimulating and managing classroom discourse so that both the students and the teacher are clearer about what is being learned; creating a classroom environment to support teaching and learning mathematics; and analysing students' learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions.

A central aim of teacher education programmes is to impart the ability of transforming disciplinary knowledge into a form of knowledge appropriate for pupils and specific to the task of teaching. These programmes hence consist of various components dealing with the multiple aspects of the knowledge teachers need to acquire to that effect, including: knowledge of mathematics, knowledge of instructional representations of mathematics suitable for pupils, knowledge about learners' cognition, knowledge of curricula, general pedagogical knowledge, and knowledge of didactics of mathematics. Underlying teacher programmes and calling for their specificity is thus a vision of teachers as professionals, the 'professionals of the pedagogical act'.

I would now like to comment on the first of these components, the acquisition of mathematical content by prospective teachers and to suggest some means of furthering the development of a conceptual understanding of mathematics in such a context. My purpose here is not to enter into a detailed description of possible courses, but rather to suggest some general guidelines — in the final part of this paper, I will indicate a few examples of mathematical topics suitable in such a context. (In addition to reports produced in various countries about mathematics teacher education, the studies of Morris (1984), although more than a decade old, still form a precious source on teacher education as seen from an international perspective.)

In many colleges or universities, the mathematical preparation offered to prospective schoolteachers can be roughly described as follows. Primary school teachers often have no mathematics courses at all to take after having completed high school. Their only post-secondary mathematical activities take place in pedagogy courses. Hence they have no occasion to develop adult-level insights into the mathematics they will be teaching to children. Secondary school teachers will quite obviously take several mathematics courses, typically in the context of a major programme in mathematics. However, one faces here the criticism made by Pólya: these courses are not explicitly intended for prospective teachers. Some will be offered to various categories of students (scientists, engineers, etc.) while others will be advanced courses for mathematics specialists. It is only in the rare case that prospective secondary school teachers will be given mathematics courses specifically designed for them. In spite of the fact that they do acquire some high-level mathematical knowledge, they have no explicit occasion for making connections with the mathematical topics for which they will be responsible in school, nor of looking at those topics from an advanced point of view *à la* Klein. This situation clearly paves the way to the 'double discontinuity' paradigm discussed by Klein (1932), as mentioned above.

A consensus has emerged to the effect that it is not by simply having primary and secondary school teachers take a greater number of traditional mathematics courses that the situation will improve. There is an urgent need for courses specifically intended for teachers and in which a different spirit is conveyed through the choice of topics, the way they are treated, the teaching methods used. As Swafford (1995), p. 161, puts it, "it is not enough to know more mathematics. Teachers must also know more about the mathematics they will teach." Such courses should thus provide prospective teachers with the opportunity to analyse school mathematics from a deeper perspective than the one which prevailed in their own schooling, giving them the ability to explicate the meanings underlying elementary mathematical topics. The ‘O-script/A-script method’ of Wittmann (this volume, pp. 539-551), can be seen as a particular instance of this teaching principle. Based on the general philosophy that the mathematical education of any category of students should reflect in a fundamental and systematic way their professional context, the approach of Wittmann allows, for instance, primary student teachers, through learning by discovery, to develop a better appreciation of reasoning as an objective in mathematics education. Wittmann also puts forward a strong plea for mathematicians not to look down on primary mathematics education.

Although it does not seem possible to speak here of a universal consensus, the idea that mathematicians should see the mathematical education of primary school teachers as part of their responsibilities clearly appears as an emerging trend. This is for instance the standpoint taken recently in the USA by the American Mathematical Society — basically a research society! — in its ‘National Policy Statement’ (AMS, 1994). Involvement of mathematics departments in primary teachers education is also one of the recommendations found in a report from the Mathematical Association of America on the mathematical preparation of teachers, ‘A Call for Change’ (see Leitzel, 1991, p. 11): “The mathematical experiences recommended for teachers at the K–4 level require that mathematics departments offer courses specifically designed for this audience.”

In order to succinctly suggest general objectives that can be attached to mathematics courses for schoolteachers education, I would like to state some standards proposed in the report ‘A Call for Change’ (Leitzel, 1991, p. 1): primary and secondary school teachers should come to view mathematics as a system of interrelated principles; to communicate mathematics accurately, both orally and in writing; to understand the elements of mathematical modelling; to understand and use calculators and computers appropriately in the teaching and learning of mathematics; and to appreciate the development of mathematics both historically and culturally.

The same report also proposes a set of standards pertaining to particular mathematical topics and related either to primary or secondary school teaching. These proposals are presented not as descriptions of specific courses, but rather as indications of content areas in very general terms. The philosophy can probably be best perceived through the following comment — which reminds one of Pólya’s statement quoted above: “The specific topics covered are not as important as *how* those topics are taught.” (Leitzel, 1991, p. 27)

Putting in practice recommendations such as those appearing in ‘A Call for Change’ may require a substantial change of mentality among many mathematicians. The design of a specific disciplinary content using as framework the needs of future teachers is far from being part of the general tradition of departments of mathematics, especially in the case of primary school teacher education. But a number of examples can be exhibited of successful accomplishments of such objectives, some having even been going on for many years.

2.5 Contexts for the implication of mathematicians in education

I would like to conclude this part of the paper by commenting on possible frameworks through which mathematicians can become involved both in school mathematics and teacher education issues. I will distinguish three different levels at which such an implication can happen: the level of the mathematician as an individual; the level of an academic unit like a mathematics department; and the level of a professional body such as a national mathematical society.

There are numerous instances of mathematicians being involved in education as a result of a personal initiative. In some cases, this educational interest will be only occasional, while in others it will be much more sustained. As indicated above, one can even easily think of quite a few famous mathematicians in this connection. In addition to Felix Klein and other mathematicians mentioned earlier, a famous example is Hans Freudenthal who, after a long and substantial career as a research mathematician, became a leader in mathematics education as a research field. He had a long-lasting influence through such accomplishments as the founding of a research institute in his country, the promotion of ICMI activities during his presidency (among which were the inception of the quadrennial International Congresses on Mathematical Education), or the creation of the research journal *Educational Studies in Mathematics*.

But such individual initiatives, however prestigious their proponents, are always at risk of having limited influence if they are not fully supported by the 'system'. A crucial aspect is thus the recognition and encouragement by the academic unit to which the individual professionally belongs (typically, in my discussion, a mathematics department). This includes, among other aspects, career issues such as promotions, tenure, etc. Sometimes the educational activities of mathematicians will even be officially presented as part of the agenda of the academic unit. A number of examples of varied depth could be given here and I will briefly present a rather unusual 'success story', the Community Teaching Fellowship Program of the University of California.

The CTF programme is an original example of a possible context for mathematicians to get seriously involved in matters of education. As the CTF programme started more than three decades ago and as it covers the eight campuses of the University of California, there have been variations in its functioning. But the basic idea at its origin was to involve mathematics graduate students (most of them in the Ph.D. programme, but also some in the M.A. programme) in providing mathematics enrichment to primary school children in minority and poverty neighbourhoods — this is a way for these students to receive financial support from the university, instead of through the regular teaching assistantship. Departing from usual remedial activities, the CTF programme concentrates on developing children's potential and building their confidence by the introduction of higher-level mathematics topics in simple and appealing ways. A graduate student participating in the programme goes to a local classroom three to five times a week to present activities in which children are encouraged to 'discover' mathematics. A premise of the programme is that the ability to use a discovery teaching method requires an ease and familiarity with mathematics that comes only through years of study, so that the average primary school teacher should not be expected to have the mathematical background necessary to teach by discovery the mathematics covered in a typical CTF class.

The CTF programme appears rather successful from various perspectives: schoolchildren seem to benefit from it, according to testimony from advisors, teachers and parents as well as attitude questionnaires; graduate students report a change in their vision of the teaching of mathematics — in a few cases, the CTF fellowship was even the beginning of a long-term career interest in mathematics education within a department of mathematics; and finally there is an impact on the mathematics departments, both from the point of view of the perception of the importance of educational matters and with respect to new collaborations established on some campuses between the department of mathematics and the school of education.

A third possible context for the involvement of mathematicians in educational issues is the general professional environment surrounding their work, in particular in connection with mathematical associations to which they may adhere. It is a most encouraging sign that in many countries — this is the case at least in the two countries I am most familiar with, Canada and the USA — professional societies of mathematicians clearly identify education as part of their responsibilities, including the education of teachers of all levels. For instance, the following can be found in a recent policy statement of the American Mathematical Society (see AMS, 1994): "The AMS will support increased participation of mathematicians in programmes for the professional development of teachers of mathematics." Numerous education sessions are now regularly organised at the annual meetings of societies of mathematicians, something almost unheard of a few decades ago. The strong opinion expressed on these matters by such a distinguished mathematician as Thurston (1990) is definitely influential in fostering evolution of mentalities.

3. MATHEMATICAL TOPICS FOR TEACHERS

In order to help prospective teachers develop the necessary deep conceptual understanding of the school mathematics content which falls under their responsibility, they should encounter in their university courses mathematical topics linked to the school curricula and presenting a good potential for instilling strong insights. I will now briefly sketch a few examples which I have successfully used with my primary and secondary school student teachers. I make no claim for high originality here, as most of these problems will probably be already familiar to many of the readers. The point I wish to bring out is the richness of the mathematical phenomena which are accessible in this context: working with prospective teachers can be fully gratifying, even from the strict point of view of the mathematical problems to be dealt with. I also want to show how some themes are appropriate for both the primary and secondary school teachers, as they can be subjected to an increasing depth of treatment. While many of the following examples could be adapted so to be presented to pupils of the appropriate level, the target audience I have in mind here is the teachers themselves.

3.1 Dilemmas of decimal developments

A ***decimal fraction*** is a fraction equivalent to one whose denominator is a power of 10. It thus corresponds to a terminating decimal development, i.e., an expansion with a finite number of non-zero digits following the decimal mark. It is natural to ask for a characterization of the decimal fractions. And the answer is remarkably simple: a fraction (in its lowest terms) is decimal if and only if the sole prime factors of its denominator are 2 and 5. Moreover the length of the expansion can be easily deduced from the prime factorisation of this denominator. The proof of these basic facts involves only the notion of prime factorisation and is readily accessible to primary school teachers.

If a fraction is not decimal, then its decimal expansion is non-terminating, but periodic: a certain sequence of digits is bound to repeat indefinitely. The expansion is then made of two parts: the ***period***, comprising the block of digits which reappear *ad infinitum*, eventually preceded by a non-repeating part, the ***pre-period***. A problem suitable for secondary school teachers is the following: given a non-decimal fraction a/b (in lowest terms), can we predict, by simple inspection of the numerator a and denominator b, the lengths of the pre-period and of the period of its decimal expansion? The answer to this question is again a simple application of elementary number theory and can be easily found in various textbooks (see for instance Rosen, 1988, Chapter 10, where the discussion is made for expansions in any base).

There are many ways of pursuing this study of decimal expansions. I indicate one, which brings into the picture more advanced results from the theory of numbers. It is striking that the periods of 1/7 and 1/17 are optimal, in the sense that all possible remainders are used before repetition — the former has period of length 6 and the latter, 16; such is not the case for 1/3, 1/11 or 1/13, which have 'short' periods. How can this phenomenon be explained? It is not difficult to show (see Rademacher and Toeplitz, 1957, Chapter 23) that in general the length of the period of a/b is a divisor of $\varphi(b)$, the Euler phi-function. The period of $1/p$ is thus optimal when it is equal to $p-1$, which happens only in the exceptional cases where 10 is a *primitive root modulo p*. For instance, it can be checked that the only primes below 100 generating optimal periods are 7, 17, 19, 23, 29, 47, 59, 61 and 97. Very little is known in general about such primes (see Hardy and Wright, 1979, Section 9.6).

3.2 A night at the hotel

The following problem is nearly a classic in the mathematics literature for primary school student teachers — see among others the version in Bell *et al.* (1976), p. 620.

> At Long Hotel, there are n rooms all located along a very long corridor and numbered consecutively from 1 to n. One night, after dinner, the n guests play the following game. The first guest runs down the corridor and opens all the doors. Then the second guest runs down the corridor and closes every second door beginning with door 2. Afterwards, the third guest changes the position of every third door beginning with door 3 (that is, the guest opens the doors that are closed and closes those that are open). In a

> similar way, the fourth guest changes the position of doors 4, 8, 12, … . This process continues until the nth guest runs down to the end of the corridor to change the position of door n. Which doors are left open and which ones are left closed at the end of the game?

Although complex at first glance, this problem obeys a remarkably simple rule: *all doors end up closed except those whose numbers are perfect squares*. This stems from the fact that at the end of the process, door d will be open or closed depending only on the parity of the number of divisors of d. The fact that the only numbers with an odd number of divisors are the perfect squares can readily be justified by primary school student teachers. This problem also provides a nice setting for exploring notions playing an important role in primary school teaching, such as (common) divisor, multiple, gcd or lcm.

Based on the 'what-if-not?' strategy, the following variant of the preceding problem was proposed by Cassidy and Hodgson (1993) for secondary school student teachers.

> The rooms of Circle Hotel are built around a circular courtyard and are numbered consecutively from 1 to n. One night, after dinner, the n guests want to play the same game as at Long Hotel. But since the hotel has the form of a circle, each of the guests could go endlessly round the corridor. So it is agreed that a guest should stop as soon as he or she changes the position of door n — which is bound to happen for every guest. Which doors are left open and which ones are left closed at the end of the game?

This problem is of the sort that responds to an inductive approach. Experimenting with a few concrete cases leads to the following observation, which is easily proved using basic notions of modular arithmetic: in a hotel with n rooms, if guest k ($k \neq n$) changes the position of a door, then so does guest $n - k$. The problem thus boils down to the single action of guest n, when n is odd, and to the combined action of guests n and $n/2$, when n is even. This directly leads to the conclusion that *for any n, all the doors are left closed except a single one; and this exceptional door is door n when n is odd, and door n/2 when n is even*.

The Circle Hotel problem can be extended in various directions. For instance, from the point of view of a given guest, one can ask how many times this guest goes round the corridor before stopping, and how many doors have been touched at the end of this action. And from the point of view of a given door, one can look for a characterization of the guests changing the position of a certain door during the process, and for the number of such guests. These questions directly relate to notions such as gcd and lcm. It can thus be shown, using Bachet-Bézout's relation expressing the gcd of two integers as a linear combination of these, that in a circular hotel with n rooms, guest k touches door d exactly when gcd (k,n) is a divisor of d.

3.3 Kaleidoscopic visions

My final example is from geometry and concerns the images that can be observed in a kaleidoscope. Ever since its invention by the Scottish physicist Sir

David Brewster in the early 19th century, the kaleidoscope has fascinated people of all ages through the richness and beauty of the pictures created by the interplay of mirrors. Using such an 'attention-catcher', teachers can bring their pupils into discovery activities about mathematical topics closely connected to central themes of the school geometry curriculum.

The understanding of the mathematical principles underlying the kaleidoscope is a challenge fully appropriate for primary school student teachers. The mastery of such a mathematical 'micro-theory' can have a positive impact on their perception of mathematics and their personal relation to it. The study of the kaleidoscopic phenomenon should go through various phases before a formal description is obtained: first looking through genuine kaleidoscopes — I always bring my small collection of kaleidoscopes into the classroom when I work this topic; then manipulation of real mirrors in a free setting (instead of fixed mirrors in a tube), using as a motif small objects or a figure drawn on a sheet of paper; and finally paper-and-pencil activities involving abstractly defined transformations, eventually performed with ruler and compass. (A further step could be the implementation of the kaleidoscopic phenomenon on the computer.) A guided exploration of elementary 'kaleidoscope geometry' can be found in Hodgson (1987).

With secondary school student teachers, the rosaces (or rose-patterns) produced by kaleidoscopes could be studied from the point of view of group theory, namely with the help of symmetry groups. A further extension would be to modify the type of mirrors 'inside' the kaleidoscope — some of the resulting kaleidoscopes have no real physical counterparts. For instance (standard) axial reflection could be replaced by central reflection (i.e., reflection in a point instead of a straight line). Or one could consider kaleidoscopes based on the geometric transformation of circular inversion. The resulting patterns can be interpreted as a new type of tessellation enjoying remarkable geometrical properties (Hodgson and Graf, 2000). Here again the computer is an exceptional tool in exploring these fictitious kaleidoscopes (see Graf and Hodgson (1998) for a discussion of such pedagogical environments).

4. CONCLUSION

While I do believe that mathematicians have a crucial contribution to bring to the mathematics education of both primary and secondary school teachers, I do not pretend that this could or should be done to the exclusion of their colleagues from schools of education. The contribution of mathematicians is different from that of mathematics educators, each having its specificity, its own aims. My point in this paper was to stress that the presence of mathematicians in the complex process of teacher preparation is not only desirable, but in fact essential.

An interesting challenge for mathematicians involved in teacher education is to identify topics and problems that can or should be worked with their students. One possible approach is to concentrate on the idea of seeing *elementary mathematics from an advanced standpoint* which I discussed in the first part of the paper: it can be truly profitable for students teachers to review from an adult perspective basic notions of mathematics they have learned in school, trying to gain a global and unifying understanding and to develop new insights. I have argued above that this constitutes the crux of the mathematical education of teachers. Another approach is to identify topics from more advanced mathematics which lend themselves to an elementary discussion; to borrow the title from a remarkable book by Rademacher (1983), one now looks at *higher mathematics from an elementary point of view*. The point here is not to trivialise mathematics, but to convey, when possible, the core ideas of a topic in elementary terms, avoiding the use of non-essential sophisticated tools. The above examples illustrate, I hope, both these approaches.

The implication of mathematicians in teacher education raises many difficult issues. One is the professional recognition of their educational work by universities or by their mathematical colleagues, as I discussed above. Another one is the reception of their efforts by professional mathematics educators. Still another is the acquisition by active mathematicians of an expertise in matters pertaining to teacher education, as they have had no formal training in mathematics education during their own graduate studies.

A major problem that arises is thus the education of mathematicians serving as teachers' educators. It is recognised that one of the main factors influencing prospective teachers is the way they are taught themselves: *what* they learn is fundamentally connected with *how* they learn it. Hence college and university mathematicians must use with student teachers the same methods these persons will be expected to use in their own classrooms: more student interaction, less lecturing and memorisation, open-ended problem-solving, etc. This might require substantial changes in the teaching habits of many faculty members.

For the involvement of the mathematics community in educational issues to be fully productive, there is a great need for a better collaboration between mathematicians in departments of mathematics, mathematics educators in schools of education and experienced mathematics teachers in the schools. It is important to tear down the 'iron curtains' which exist in many countries between the various educational levels, and to invigorate the communication channels between them. For instance the question of the 'dialogue' of mathematics education researchers "with other scientific communities, in particular the mathematics research community" (Sierpinska and Kilpatrick, 1998, p. x) was one of the issues raised at the outset of the recent ICMI Study on research in mathematics education. While it is important for educators to keep informed on the evolution of contemporary mathematics, there is also an urgent need for mathematicians to become better aware of the researches of their colleagues in education and of their potential impact both on their own teaching and on school teaching.

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