Statistics & Probability Letters 8 (1989) 207–211
North-Holland August 1989

A CHARACTERIZATION OF GUMBEL'S FAMILY OF EXTREME VALUE DISTRIBUTIONS

Christian GENEST and Louis-Paul RIVEST

Département de mathématiques et de statistique, Université Laval, Cité universitaire, Québec, Canada G1K 7P4

Received August 1988
Revised October 1988

Abstract: In this note, a family of multivariate extremal distributions proposed by Gumbel (1960) is characterized among those whose dependence function is an Archimedean copula. The domains of attraction of Gumbel's distributions are also determined within this class.

Keywords: Archimedean copula, dependence function, domain of attraction, extreme value distribution, Gumbel's family, regular variation.

1. Introduction

Let $F_1(x_1), \ldots, F_k(x_k)$ represent $k \geqslant 2$ univariate max extreme value distributions. Gumbel (1960) was possibly the first to observe that functions $H_m(x_1, \ldots, x_k)$ defined for real $m \geqslant 1$ by

$$\{-\log H_m(x_1, \ldots, x_k)\}^m = \sum_{i=1}^{k} \{-\log F_i(x_i)\}^m$$

are multivariate extremal distributions. This system of distributions, which includes independence when $m = 1$, is well known. It is mentioned by Berman (1961) and by Marshall and Olkin (1983), among others, in the context of extreme value theory.

The purpose of this note is to characterize Gumbel's family of multivariate extreme value distributions among those whose dependence function is an Archimedean copula. Sibuya (1960) defines the dependence function of an arbitrary distribution F with continuous marginals $F_1, \ldots, F_k$ as

$$C(x_1, \ldots, x_k) = F\{F_1^{-1}(x_1), \ldots, F_k^{-1}(x_k)\}. \quad (1)$$

In the terminology of Schweizer and Sklar (1983), $C(x_1, \ldots, x_k)$ is a k-dimensional copula, i.e., a distribution with uniform marginals on (0, 1). A copula is said to be Archimedean if it can be expressed in the form

$$C(x_1, \ldots, x_k) = \phi^{-1}\left\{\sum_{i=1}^{k} \phi(x_i)\right\}, \quad (2)$$

where $\phi(t)$ is a function defined on (0, 1] in such a way that $\phi(1) = 0$ and

$$(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \phi^{-1}(t) \geqslant 0, \quad 1 \leqslant i \leqslant k.$$

Note that ϕ is unique up to a (positive) multiplicative constant and that when $k = 2$, the conditions on ϕ are equivalent to saying that $K(t) = 1 - \phi^{-1}(t)$ is the distribution function of a unimodal random variable with mode at zero.

Since a multivariate distribution F is uniquely determined by its marginals, F_i, and its dependence function (1), it is possible to investigate the convergence of F to an extreme value distribution by considering separately the behavior of its marginals and that of the corresponding copula. Thus if F_i belongs to the domain of attraction of F_i^*, $1 \leqslant i \leqslant k$, and if

$$\lim_{n\to\infty} C^n(x_1^{1/n},\ldots,x_k^{1/n}) = C^*(x_1,\ldots,x_k)$$

for all $0 < x_1,\ldots,x_k < 1$, then according to Theorem 5.2.3 in Galambos (1978), F belongs to the domain of attraction of

$$F^*(x_1,\ldots,x_k) = C^*\{F_1^*(x_1),\ldots,F_k^*(x_k)\}.$$

When the limiting distribution, F^*, belongs to Gumbel's family, so that $F^* = H_m$ for some $m \geqslant 1$, we observe that the corresponding dependence function, $C^* = G_m$, is an Archimedean copula of the form (2) with

$$\phi(t) = \gamma_m(t) \equiv \{-\log(t)\}^m, \quad 0 < t \leqslant 1.$$

It is thus natural to ask whether it is possible to generate max extreme value distributions from other Archimedean copulas. It is this question which originally motivated the present research.

Using general results of Genest and MacKay (1986) on Archimedean copulas, the following statements will be proved in Section 2, respectively Section 3.

Statement A. *Gumbel's distributions are the only max extreme value distributions whose dependence function is an Archimedean copula.*

Statement B. *Distributions whose dependence function is an Archimedean copula of the form* (2) *belong to the max domain of attraction of Gumbel's distribution* H_m (*with appropriate marginals*) *if and only if* $\lambda'(1)$ *exists and equals* $1/m > 0$, *where* $\lambda(t) = \phi(t)/\phi'(t)$, $0 < t < 1$.

In Section 4, these results will be related to classical theorems from the theory of univariate extremes. As the proofs for the case $k = 2$ generalize immediately to higher dimensions, we will restrict ourselves to this case for convenience.

2. A characterization of Gumbel's family

Let $(X_1, Y_1),\ldots,(X_n, Y_n)$ be a random sample from a bivariate Archimedean copula of the form (2). Define $X = \max(X_i)$ and $Y = \max(Y_i)$. Then

$$\begin{aligned}\mathrm{pr}(X \leqslant x, Y \leqslant y) &= [\phi^{-1}\{\phi(x) + \phi(y)\}]^n \\ &= \phi_n^{-1}\{\phi_n(x^n) + \phi_n(y^n)\}\end{aligned}$$

with $\phi_n(t) = \phi(t^{1/n})$, $0 < t \leqslant 1$, so that the dependence function of (X, Y),

$$C_n(x, y) = \phi_n^{-1}\{\phi_n(x) + \phi_n(y)\},$$

is again an Archimedean copula.

Setting $\lambda(t) = \phi(t)/\phi'(t)$, we get

$$\begin{aligned}\phi_n(t)/\phi_n'(t) &= nt^{1-1/n}\phi(t^{1/n})/\phi'(t^{1/n}) \\ &= t^{1-1/n}\left(\frac{t^{1/n}-1}{1/n}\right)\left(\frac{\lambda(t^{1/n})}{t^{1/n}-1}\right).\end{aligned}$$

Thus $\phi_n(t)/\phi_n'(t)$ has a limit as n tends to infinity if and only if $\lambda'(1)$ exists, in which case

$$\lim_{n\to\infty} \phi_n(t)/\phi_n'(t) = \lambda'(1)t\log(t), \quad 0 < t \leqslant 1. \tag{3}$$

Now if C is the dependence function of an extremal distribution, we know that

$$C_n(x, y) = C(x, y) \quad \text{for all } 0 < x, y < 1.$$

This condition, which Leadbetter and Rootzén (1988) term max-stability, derives from the fact that an extreme value distribution always belongs to its own domain of attraction. Appealing to Theorem 5.4.8 in Schweizer and Sklar (1983), we deduce that

$$\phi(t)/\phi'(t) = \phi_n(t)/\phi_n'(t), \quad 0 < t \leqslant 1, \tag{4}$$

for all $n \geqslant 1$. It is then immediate from (3) and (4) that $\lambda'(1) > 0$ exists and that

$$\phi(t)/\phi'(t) = t\log(t)/m, \quad 0 < t \leqslant 1,$$

where $m = 1/\lambda'(1) \geqslant 1$. We conclude that $\phi(t) = \{-\log(t)\}^{1/m}$ up to a multiplicative constant, i.e., $C(x, y) = G_m(x, y)$ for some $m \geqslant 1$. This completes the proof of Statement A.

3. Domains of attraction of Gumbel's distributions

Once again, consider a bivariate Archimedean copula

$$C(x, y) = \phi^{-1}\{\phi(x) + \phi(y)\}$$

generated by a decreasing, convex function $\phi(t)$ with $\phi(1) = 0$, and let $\lambda(t) = \phi(t)/\phi'(t)$, $0 < t < 1$.

Using the same notation as in Section 2, we must show that

$$\lim_{n\to\infty} C_n(x, y) = \gamma_m^{-1}\{\gamma_m(x) + \gamma_m(y)\} \tag{5}$$

for all $0 < x, y < 1$ if and only if $\lambda'(1)$ exists and equals $1/m > 0$. To this end, we will make use of the following extension of Proposition 4.2 in Genest and MacKay (1986), stated here without proof.

Proposition. *Let $H_n(x, y)$ be an Archimedean copula generated by ϕ_n and let $H(x, y)$ be another Archimedean copula generated by ϕ. The following statements are equivalent*:

(i) *$H_n(x, y)$ converges to $H(x, y)$ for all $0 < x, y < 1$*;

(ii) *$\phi_n(s)/\phi_n'(t)$ converges to $\phi(s)/\phi'(t)$ for all $0 < s \leqslant 1$ and all values of $0 < t \leqslant 1$ where ϕ' is continuous*;

(iii) *There exists a sequence $a_1, a_2, \ldots,$ of strictly positive numbers such that $a_n\phi_n(s)$ converges to $\phi(s)$ for all $0 < s \leqslant 1$.*

The necessity of the condition on $\lambda'(1)$ is immediate from (5), since part (ii) of the above proposition implies that

$$\lim_{n\to\infty} \phi_n(t)/\phi_n'(t) = \log(t)/m$$

for all $0 < t < 1$. To show sufficiency, assume that $\lambda'(1) > 0$ exists and set $m = 1/\lambda'(1)$. In view of (3) and part (ii) of the above proposition, it will suffice to show that $\phi_n(s)/\phi_n(t)$ converges to $\{\log(s)/\log(t)\}^m$ as n increases indefinitely or, equivalently, that

$$\lim_{n\to\infty} \log\{\phi(s^{1/n})/\phi(t^{1/n})\} = m \log\{\log(s)/\log(t)\} \tag{6}$$

for all $0 < s, t < 1$. To do this, first rewrite the expression on the left-hand side of (6) as

$$\int_{t^{1/n}}^{s^{1/n}} 1/\lambda(x)\,\mathrm{d}x = \log \phi(s^{1/n}) - \log \phi(t^{1/n}).$$

Making the one-to-one transformation $y = n\log(x)$ under the integral sign, we find

$$\int_{t^{1/n}}^{s^{1/n}} 1/\lambda(x)\,\mathrm{d}x = \int_{\log(t)}^{\log(s)} \frac{\mathrm{e}^{y/n}}{n\lambda(\mathrm{e}^{y/n})}\,\mathrm{d}y.$$

Since $\lambda'(1) = 1/m$ exists, we can now use the fact that $(\mathrm{e}^{y/n} - 1)/\lambda(\mathrm{e}^{y/n})$ approaches m asymptotically to show that

$$\lim_{n\to\infty} \int_{t^{1/n}}^{s^{1/n}} 1/\lambda(x)\,\mathrm{d}x = \lim_{n\to\infty} m\int_{\log(t)}^{\log(s)} \frac{\mathrm{e}^{y/n}}{n(\mathrm{e}^{y/n} - 1)}\,\mathrm{d}y. \tag{7}$$

Finally, the limit on the right-hand side of (7) is equal to

$$\lim_{n\to\infty} m \log\left(\frac{1 - \mathrm{e}^{\log(s)/n}}{1 - \mathrm{e}^{\log(t)/n}}\right) = m \log\{\log(s)/\log(t)\}$$

by L'Hospital's rule, and the proof of Statement B is complete.

Illustrations. Several well-known systems of bivariate distributions whose dependence function is an Archimedean copula belong to the domain of attraction of independence. This is the case for the families introduced by Ali, Mikhail and Haq (1978), Cook and Johnson (1981) and Frank (1979) [1], among others. An Archimedean copula whose limiting max extreme value distribution is Gumbel's distribution with parameter $m > 1$ can be obtained by taking $\phi_m(t) = (1 - t)^m$, $0 < t \leqslant 1$. This function gives rise to a bivariate uniform distribution which has a mass of $1/m$ accumulated on the curve $(1 - x)^m + (1 - y)^m = 1$.

Remark. Since ϕ is convex and $\phi(1) = 0$ by construction, it is easy to see that for given $0 < t < 1$, the line $y = \phi(t) + \phi'(t)(x - t)$ must cross the x-axis between 0 and 1, so that $t - 1 \leqslant \lambda(t) \leqslant 0$ always. Thus it could happen that $\lambda'(1)$ exists and equals 0. In that case, we can call upon Proposition 4.3 of Genest and MacKay (1986) to conclude directly that the limiting distribution is the Fréchet upper bound, viz. $\min\{F_1^*(x), F_2^*(y)\}$. The Archimedean copula generated by $\phi(t) = \exp\{2/(t - 1)\}$ provides an example.

[1] The statistical properties of this particular system of joint distributions have been investigated by Nelsen (1986) and Genest (1987).

4. Connection with univariate results

It is instructive to relate Statement B to classical characterizations of univariate max domains of attraction. By part (iii) of the above proposition, we know that an Archimedean copula of the form (2) converges to G_m, the dependence function of Gumbel's distribution H_m, if and only if

$$\lim_{n\to\infty} a_n\phi(t^{1/n}) = \gamma_m(t), \quad 0 < t \leqslant 1,$$

for some sequence of positive numbers a_1, a_2, ... Taking inverses and setting $\rho = -1/m$, we see that

$$\lim_{n\to\infty} \left\{\phi^{-1}(1/a_n s)\right\}^n = \exp(-s^\rho) \tag{8}$$

must hold true for all values of $s > 0$. Equation (8) can be recast as follows in terms of a one-dimensional extreme value theorem.

Consider a random sample $Z_1, \ldots, Z_n$ from $L(s) = \phi^{-1}(1/s)$. If $M_n = \max(Z_1, \ldots, Z_n)$, then

$$\mathrm{pr}(M_n \leqslant s) = L^n(s),$$

so that

$$\begin{aligned}\mathrm{pr}(M_n/a_n \leqslant s) &= L^n(a_n s)\\ &= \left\{\phi^{-1}(1/a_n s)\right\}^n, \quad s > 0. \end{aligned}\tag{9}$$

In view of (8), this implies that L belongs to the max domain of attraction of $\exp(-s^\rho)$. According to a classical result of R.A. Fisher and B.V. Gnedenko (cf. Theorem 2.4.3, part (i), in Galambos, 1978), this occurs if and only if $1 - L(1/t) = 1 - \phi^{-1}(t)$ varies regularly with exponent $\rho = -1/m < 0$. In other words, we have shown that $\lambda'(1)$ exists and is strictly positive if and only if the distribution function $K(t) = 1 - \phi^{-1}(t)$ is of regular variation at the origin.

Statement B can also be recast as follows in terms of an older limiting result due to R. von Mises. Theorem 2.7.1 in Galambos (1978) states that a sufficient condition for $L(s) = \phi^{-1}(1/s)$ to belong to the domain of attraction of $\exp(-s^\rho)$ is that

$$\lim_{s\to\infty} sl(s)/(1 - L(s)) = -\rho = 1/m > 0, \tag{10}$$

where $l(s)$ denotes the probability density function associated with L. In the context of the present paper, equation (10) translates into

$$\lim_{t\to 1} \lambda(t)/(t-1) = \lambda'(1) = 1/m,$$

since $\lambda(1) = 0$ by definition. In our case, it turns out that the condition on L is also necessary, because of the particular nature of these distributions.

These relations between univariate results on the one hand, and our bivariate characterization of the limiting behavior of maxima for Archimedean copulas on the other hand, do not really come as a surprise. As we pointed out in the introduction, there is a one-to-one correspondence between bivariate distributions generated by convex functions $\phi(t)$, as in (2), and the univariate distribution functions $K(t) = 1 - \phi^{-1}(t)$ of unimodal random variables with mode at zero. Because of this special feature, we believe Archimedean copulas are well suited as a theoretical model for the statistical investigation of multivariate data.

References

Ali, M.M., N.N. Mikhail and M.S. Haq (1978), A class of bivariate distributions including the bivariate logistic, *J. Multivariate Anal.* **8**, 405–412.

Berman, S.M. (1961), Convergence to bivariate limiting extreme value distributions, *Ann. Inst. Statist. Math.* **13**, 217–223.

Cook, R.D. and M.E. Johnson (1981), A family of distributions for modelling non-elliptically symmetric multivariate data, *J. Roy. Statist. Soc. Ser. B* **43**, 210–218.

Frank, M.J. (1979), On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$, *Aequationes Math.* **19**, 194–226.

Galambos, J. (1978), *The Asymptotic Theory of Extreme Order Statistics* (Wiley, New York).

Genest, C. (1987), Frank's family of bivariate distributions, *Biometrika* **74**, 549–555.

Genest, C. and R.J. MacKay (1986), Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données, *Rev. Canad. Statist.* **14**, 145–159.

Gumbel, E.J. (1960), Distributions des valeurs extrêmes en plusieurs dimensions, *Publ. Inst. Statist. Univ. Paris* **9**, 171–173.

Leadbetter, M.R. and H. Rootzén (1988), Extremal theory for stochastic processes, *Ann. Probab.* **16**, 431–478.

Marshall, A.W. and I. Olkin (1983), Domains of attraction of multivariate extreme value distributions, *Ann. Probab.* **11**, 168–177.

Nelsen, R.B. (1986), Properties of a one-parameter family of bivariate distributions with specified marginals, *Comm. Statist.* **15**, 3277–3285.

Schweizer, B. and A. Sklar (1983), *Probabilistic Metric Spaces* (North-Holland, Amsterdam).

Sibuya, M. (1960), Bivariate extreme statistics, *Ann. Inst. Statist. Math.* **11**, 195–210.