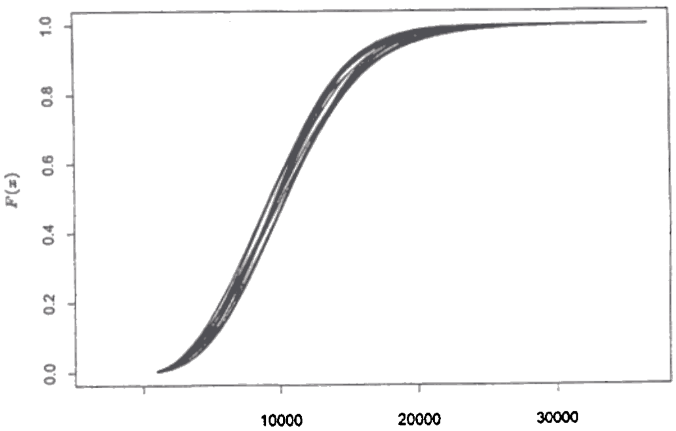
2. The GB2 model, however, is not suitable for pricing or for reinsurance since it gives a lower value of x than the gamma model for almost all values of p.
3. The mixture of two-gamma models, after applying the lemma below, becomes a four-parameter model. Its log-likelihood is marginally better than that of GB2, and we expect its confidence region is about the same width as the GB2.

Figure 4
$F(x|a, b, p, q)$ with (a, b, p, q) Running through Its 95% Confidence Region



Mixture of Linear Exponential Objects

For both normal distribution and gamma distribution, the mean of the MLE-fitted model must match the empirical mean. For the normal, this result is found in any Course 110 textbook. For the gamma, or mixture of gamma and normal, it's not difficult to prove.

Lemma (Mixture of Exponential Family with a Linear Term)

If X has the density function

$$f(x) = h(\alpha, x)c(\alpha, \beta) \exp(-\beta x) \tag{1}$$

then

$$E_f(X) = \int x f(x)\, dx = \frac{\frac{\partial c}{\partial \beta}(\alpha, \beta)}{c(\alpha, \beta)} \tag{2}$$

and the mean of the maximum likelihood estimated density function of the form (1) fitted to observed data would match the empirical mean:

$$E_{\hat{f}}(X) = \frac{\frac{\partial c}{\partial \beta}(\hat{\alpha}, \hat{\beta})}{c(\hat{\alpha}, \hat{\beta})} = \frac{\sum x}{n}. \tag{3}$$

And a similar result holds if X is a mixture of two densities satisfying Equation (1)

$$f(x) = pf_1(x) + (1 - p)f_2(x)$$
$$ph_1(\alpha_1, x)c_1(\alpha_1, \beta_1) \exp(-\beta_1 x)$$
$$+ (1 - p)h_2(\alpha_2, x)c_2(\alpha_2, \beta_2) \exp(-\beta_2 x). \tag{4}$$

Then the formula mean of the maximum likelihood estimated density function of the form (4) fitted to observed data $\{x_1, \ldots, x_n\}$ would match the empirical mean:

$$E_f(X) = \frac{\frac{\partial c}{\partial \beta}(\alpha, \beta)}{c(\alpha, \beta)} = \frac{\sum x}{n}.$$

This lemma says, for example, that a gamma mixed with normal fitted by MLE will match the empirical mean.

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"Understanding Relationships Using Copulas," by Edward Frees and Emiliano Valdez, January 1998

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This article aimed to introduce actuaries to "copulas"—that is, distributions whose univariate marginals are uniform—as a tool for understanding relationships among multivariate outcomes. Through its limpid exposition of some of the recent developments in

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statistical modeling with copulas, this work and its substantial annotated bibliography should contribute much to the dissemination of these techniques in actuarial circles.

In their paper, the authors explain how it is possible, using copulas, to dissociate the choice of marginal distributions from that of a model describing the dependence between pairs of correlated variables such as the economic loss and the medical indemnity component of a disability policy. Part of their discussion revolves around the issue of identifying an appropriate family of copulas and the estimation of its parameters from a multivariate random sample. In Section 4.2.1, they show specifically how an indemnity payment X_1 (loss) and an allocated loss adjustment expense (ALAE) X_2 could profitably be modeled from a random sample of 1,500 liability claims. The copula they select is of the Archimedean variety, which means that for all $0 \leq u, v \leq 1$, it may be written in the form

$$C(u, v) = \phi^{-1}\{\phi(u) + \phi(v)\} \quad (1)$$

for some decreasing, convex function ϕ: $(0,1] \rightarrow [0, \infty)$ such that $\phi(1) = 0$, with the convention that $\phi^{-1}(t) = 0$ for all $t \geq \lim_{s \downarrow 0} \phi(s) \equiv \phi(0)$. In that context, they find that the generator $\phi(t) = \log^{\alpha}(1/t)$ of Gumbel's family of copulas provides an adequate fit, given a suitable choice of parameter $\alpha \geq 1$.

This discussion expands on the issue of selecting an appropriate copula for modeling purposes. Cast in terms of the above example, the question to be addressed is whether it is possible to improve on Gumbel's family as a model for describing the relationship between variables loss and ALAE. Techniques for imbedding copulas of the form (1) in larger models of exchangeable and nonexchangeable variables are introduced, and the dataset considered by Frees and Valdez is used to illustrate how such extensions can yield improvements in the fit of a model. Although the proposed methodology does not lead to a better solution in this particular application, it should prove useful in a variety of contexts. Finally, the attention of the actuarial community is directed to another potentially useful collection of bivariate copulas whereof Gumbel's family is a distinguished subset: the class of extreme value copulas.

Generating Archimedean Copula Models

As pointed out by Frees and Valdez, and several others before them, many well-known systems of bivariate distributions have underlying copulas of the form (1). These Archimedean copulas (Genest and MacKay 1986) provide a host of models that are versatile in terms of both the nature and strength of the association they induce between the variables. It is not surprising therefore that they have been used successfully in a number of data-modeling contexts, especially in connection with the notion of "frailty" (Clayton 1978, Oakes 1989, Zheng and Klein 1995, Bandeen-Roche and Liang 1996, Day, Bryant and Lefkopoulou 1997).

Because copulas characterize the dependence structure of a random vector once the effect of the marginals has been factored out, identifying and fitting a copula to data poses special difficulties. The families of Clayton (1978), Gumbel (1960), and Frank (1979), for example, provide handy, one-parameter representations of the association between variables X_i with marginal distribution functions F_i. However, they are unsuitable in those situations in which the joint dependence structure of the uniform random variables $F_i(X_i)$ is not exchangeable, because Archimedean copulas in general are symmetric in their arguments. Unfortunately, diagnostic procedures that can help delineate circumstances where model (1) is adequate have yet to be analyzed.

Starting from the assumption that the Archimedean dependence structure is appropriate in a bivariate context, Frees and Valdez explain how the choice of the generator can be made, using the technique developed by Genest and Rivest (1993). Given observations from a random pair (X_1, X_2) with distribution H, this procedure relies on the estimation of the univariate distribution function $K(v)$ associated with the "probability integral transformation," $V = H(X_1, X_2)$. For each parametric Archimedean structure C_γ that is envisaged, an estimator $\hat{\gamma}$ of γ is obtained, and the $K_{\hat{\gamma}}$'s are compared, graphically or otherwise, to a nonparametric estimator of K, whose stochastic behavior as a process has recently been studied by Barbe, Genest, Ghoudi and Rémillard (1996).

Having identified an Archimedean family of copulas that provides a reasonable fit to a bivariate random sample, are there simple ways of generating alternative models that might improve this fit? One suggestion consists of enlarging the selected family through various combinations of the following rules, which can be used to construct nested classes of Archimedean generators. For ease of reference, this collection of rules is presented as a proposition, whose proof is straightforward and left to the reader.

Proposition 1

Suppose that ϕ is the generator of a bivariate Archimedean copula. In other words, assume that ϕ: $(0,1]$

$\rightarrow [0, \infty)$ is a decreasing, convex function such that $\phi(1) = 0$.

(i) (*right composition*) If f: $[0, 1] \rightarrow [0, 1]$ is an increasing, concave bijection, then $\phi \circ f(t) = \phi\{f(t)\}$ generates a bivariate Archimedean copula.

(ii) (*left composition*) If f: $[0, \infty) \rightarrow [0, \infty)$ is an increasing, convex function such that $f(0) = 0$, then $f \circ \phi$ generates a bivariate Archimedean copula.

(iii) (*scaling*) If $0 < \alpha < 1$, then $\phi_\alpha(t) = \phi(\alpha t) - \phi(\alpha)$ generates a bivariate Archimedean copula.

(iv) (*composition via exponentiation*) If $(\phi')^2 \le \phi''$ and if ψ generates a bivariate Archimedean copula, then $\psi(e^{-\phi})$ also generates a bivariate Archimedean copula.

(v) (*linear combination*) If α and β are positive reals and if ψ generates a bivariate Archimedean copula, then $\alpha\psi + \beta\phi$ also generates a bivariate Archimedean copula.

To illustrate these composition rules, define $\phi(t) = \log(1/t)$ for all $0 < t \le 1$ and consider the function $f_\alpha(t) = (e^{\alpha t} - 1)/\alpha$ on $[0, \infty)$, which satisfies the requirements of part (ii) of the proposition for arbitrary $\alpha > 0$. Then

$$\phi_\alpha(t) = f_\alpha\{\phi(t)\} = (t^{-\alpha} - 1)/\alpha, \qquad 0 < t \le 1$$

is immediately recognized as the generator of Clayton's family of bivariate copulas. Frank's and many other classical systems of bivariate Archimedean distributions can be recovered in this fashion, often with the generator $\log(1/t)$ of the independence copula as a starting point. Special cases of rules (i) and (ii) can be found in the work of Oakes (1994) and Nelsen (1997). For some examples of Archimedean (as well as non-Archimedean) copulas and for additional ways of combining them to build new bivariate and multivariate distributions, see for example Joe (1993) or Joe and Hu (1996).

The usefulness of Proposition 1 is best demonstrated in a data analytic context. In their paper, for example, Frees and Valdez envisage Clayton's, Frank's, and Gumbel's bivariate copulas for modeling the loss and expense components of an insurance company's indemnity claims. As explained below, a specific combination of composition rules (i) and (iii) makes it possible to include these three families as special cases of a single system of bivariate Archimedean copulas, whose generator is given by

$$\phi_{\alpha,\beta,\gamma}(t) = \log\left\{\frac{1 - (1 - \gamma)^\beta}{1 - (1 - \gamma t^\alpha)^\beta}\right\}, \qquad 0 < t \le 1 \tag{2}$$

with arbitrary $\alpha > 0$, $\beta > 1$ and $0 < \gamma < 1$.

The interest of this enlarged model for inferential purposes is twofold. On one hand, it may sometimes provide a significantly better fit of the data than any submodel. On the other, it suggests a simple statistical procedure for choosing between the one-parameter models: only that which is "closest" to the full model, in some sense, need be chosen. Tests of significance of the appropriate parameters can be used to assist in this choice.

A similar strategy can be designed to handle situations in which nonexchangeability is suspected between the $F_i(X_i)$'s. It is briefly described in the next section, and a concrete application is given in the following section using the liability data of Frees and Valdez. The following paragraphs substantiate some of the above claims concerning the three-parameter, bivariate Archimedean family of copulas generated by (2).

To check that $\phi_{\alpha,\beta,\gamma}$ is a valid generator of bivariate Archimedean copulas, start from $\phi(t) = \log(1/t)$, $0 < t \le 1$, and use the fact that $f_\beta(t) = 1 - (1 - t)^\beta$ is increasing and concave on $[0, 1]$ for $\beta > 1$ to conclude from (i) that $\phi_1(t) = -\log\{1 - (1 - t)^\beta\}$ generates a bivariate Archimedean copula. Next, apply rule (iii) to see that $\phi_2(t) = \phi_1(\gamma t) - \phi_1(\gamma)$ also generates a bivariate Archimedean copula. Finally, observe that since t^α is a concave, increasing function on $[0, 1]$ for $0 < \alpha < 1$, $\phi_{\alpha,\beta,\gamma}(t) = \phi_2(t^\alpha)$ is indeed a bivariate Archimedean copula generator because of rule (i). Note that function (2) is actually convex and decreasing even when $\alpha \ge 1$.

The parameter values of (2) that yield the Clayton, Frank and Gumbel families of copulas are given in Table 1, in which the notation $o(x)$ stands for a function $f(x)$ satisfying $f(x)/x \rightarrow 0$ as $x \rightarrow 0$. Note in particular that Clayton's and Gumbel's families of bivariate copulas obtain when α and $1 - \gamma$ are both going to zero. If α converges to zero faster than $1 - \gamma$, Clayton's system is actually the right limit, while when $1 - \gamma$ is going to zero faster than α, the limit is a Gumbel copula.

Table 1
Parameter Values of Archimedean Generator (2) Yielding Clayton's, Frank's, and Gumbel's Families with Dependence Parameter δ

Family	α	β	γ
Clayton	$o(1-\gamma)$	$(1-\gamma)\delta/\alpha$	$\uparrow 1$
Frank	1	$\uparrow \infty$	$-\delta/\beta$
Gumbel	$\downarrow 0$	δ	$1 - o(\alpha)$

Generating Asymmetric Copulas

All bivariate copulas C in the Archimedean class satisfy the exchangeability condition $C(u, v) = C(v, u)$ on their domain. If a family of copulas of form (1) is envisaged as a model in a situation in which the appropriateness of this symmetry condition is doubtful, one may wish to enlarge the system to include nonexchangeable models. The following proposition shows how this can be done and paves the way to a simple test of nonexchangeability, which is illustrated in the next section.

Proposition 2

Let C be an exchangeable bivariate copula. A family of nonexchangeable bivariate copulas $C_{\kappa,\lambda}$ with parameters $0 < \kappa, \lambda < 1$ that includes C as a limiting case is defined by

$$C_{\kappa,\lambda}(u, v) = u^{1-\kappa}v^{1-\lambda}C(u^{\kappa}, v^{\lambda}), \qquad 0 \le u, v \le 1$$

This mechanism for generating asymmetric copulas was first studied by Khoudraji (1995) in an unpublished doctoral dissertation prepared under the supervision of the first and third authors. An interesting property of this nonexchangeable model is that it is easy to generate random variates distributed according to $C_{\kappa,\lambda}$. Indeed, if the pair (U_1, V_1) is drawn from copula $C(u, v)$ and if U_2 and V_2 are independent observations from a uniform distribution on the interval $[0, 1]$, then $C_{\kappa,\lambda}(u, v)$ is the joint distribution of

$$U = \max\{U_1^{1/\kappa}, U_2^{1/(1-\kappa)}\}, \; V = \max\{V_1^{1/\lambda}, V_2^{1/(1-\lambda)}\}.$$

The verification of this assertion also constitutes a proof that the function $C_{\kappa,\lambda}$ defined in Proposition 2 is always a bivariate copula. A natural extension of this result is the case in which the pair (U_2, V_2) is distributed as a copula $D(u, v)$, which may be different from independence. The details are given in Chapter 4 of Khoudraji (1995).

Seeking Improvements in the Fit of a Model

To illustrate concretely how Propositions 1 and 2 can help enhance the fit provided by an Archimedean copula model, it is convenient to reconsider the loss and ALAE data of Frees and Valdez. Specifically, this section investigates whether a better model than Gumbel's can be found for these data, either using copula

$$C(u, v) = \phi_{\alpha,\beta,\gamma}^{-1}\{\phi_{\alpha,\beta,\gamma}(u) + \phi_{\alpha,\beta,\gamma}(v)\} \qquad (3)$$

or an asymmetric copula constructed by the method described in the previous section.

In their paper, Frees and Valdez explain how maximum-likelihood-based computer algorithms can assist in estimating simultaneously the parameters associated with the marginal distributions of a random pair (X_1, X_2) and those which correspond to the family of copulas selected as a model for dependence. Applying this procedure to the three-parameter copula (3) and Pareto marginals, say, would thus require the estimation of seven parameters altogether, nine if the copula were also expanded via Proposition 2 to check for asymmetry.

While this approach is sound and straightforward to implement, it might yield inappropriate values of the dependence parameters α, β and γ (and eventually κ and λ) if the parametric models chosen for X_1 and X_2 turned out to be incorrect. There is no serious reason to doubt this choice in the present case, but as a general precaution, one may wish to ensure that uncertainty about the marginals does not affect unduly the parameter estimates of the copula model. In other words, a parametric estimation procedure that is robust to the choice of marginal distributions is called for.

A margin-free parameter estimation procedure for copulas has been described in broad, nontechnical terms by Oakes (1994). This semiparametric technique was subsequently developed and studied in Genest, Ghoudi and Rivest (1995) and adapted to the case of censoring by Shih and Louis (1995). In that pseudo-likelihood-based approach, the contribution of an uncensored individual bivariate data point to the likelihood is, in the notation of Frees and Valdez,

$$C_{12}\{F_{1n}(x_1), F_{2n}(x_2)\},$$

where C_{12} is the mixed partial derivative of the specified parametric copula, and $F_{in}(x_i)$ stands for the empirical distribution function of X_i, $i = 1, 2$. In the application at hand, $F_{1n}(x)$ and $F_{2n}(x)$ would thus be the proportions of claims for which loss X_1 and ALAE X_2 are less than or equal to x, respectively.

When the copula C considered involves three parameters, as in model (3), computing the mixed partial derivative C_{12} and maximizing the pseudo-likelihood may seem like an insurmountable task. Fortunately, there is no need to have a closed analytical expression for C_{12} to produce numerical estimates of the dependence parameters. Using a software for symbolic manipulations, such as Maple, one can feed in $\phi_{\alpha,\beta,\gamma}$ and its inverse and get, as an output, subroutines for the numerical evaluation of C and C_{12}. These subroutines can then be linked to a general-purpose maximization program that finds the pseudo-likelihood estimates.

The calculations presented below were obtained with a Fortran program developed by S. G. Nash from George Mason University. Formulas for calculating the standard errors of the estimates are given in Genest, Ghoudi and Rivest (1995). The slight censoring in X_1 was ignored in the calculations, because the numerical results presented by Frees and Valdez suggest that this has a negligible impact on the estimates. Alternatively, the pseudo-likelihood procedure of Shih and Louis (1995) could be implemented to account for the censoring in X_1.

The semiparametric parameter estimates for Clayton's, Frank's, and Gumbel's families of copulas are given in Table 2, along with those of the three parameters of the Archimedean model generated by (2). The estimates obtained for Frank's and Gumbel's models are very close to those reported by Frees and Valdez (1998). This provides indirect evidence that their choice of Pareto distributions for the marginals is adequate.

As explained earlier, the graphical technique suggested by Genest and Rivest (1993) can be used to investigate the fit of the four Archimedean models at hand. However, the plots obtained with parameters estimated from the pseudo-likelihood tend to be more sensitive to model misspecification than those constructed with estimates derived from Kendall's tau. Alternatively, the value of the maximized pseudo-log-likelihood may be used, formally or informally, to judge the models' relative merits.

In Table 2, $1 - \hat{\gamma}$ is much smaller than $\hat{\alpha}$, and neither parameter is significantly different from 0. As mentioned earlier, this is a situation in which Gumbel's family is indicated. The small difference observed between the maximized pseudo-log-likelihoods for the three-parameter model and Gumbel's sub-model confirms that the latter fits well.

Using Gumbel's copula as a starting point, we may also wish to investigate the issue of asymmetry. This can be done easily with the technique presented above. The maximum pseudo-log-likelihood is 207.3 and the parameter estimates are $\hat{\alpha} = 1.47$ (s. e. = 0.07), $\hat{\kappa} = 0.94$ (s. e. = 0.09), $\hat{\lambda} = 1$. Because $\hat{\kappa}$ is not significantly different from 1, it appears that in this case, the introduction of asymmetry in Gumbel's copula does not improve the fit.

Extreme Value Copulas

Whenever Gumbel's family of copulas seems adequate for a bivariate dataset, one may also wish to check whether a better fit could be achieved by another system in the maximum extreme value class. These are copulas that can be expressed as

$$C_A(u, v) = \exp\left[\log(uv) A\left\{\frac{\log(u)}{\log(uv)}\right\}\right], \quad 0 \le u, v \le 1 \quad (4)$$

in terms of a dependence function A: $[0, 1] \to [1/2, 1]$, which is convex and verifies $A(t) \ge \max(t, 1 - t)$ for all $0 \le t \le 1$. It is easy to see that the choice

$$A(t) = \{t^\alpha + (1 - t)^\alpha\}^{1/\alpha}, \qquad 0 \le t \le 1$$

corresponds to Gumbel's family of copulas, which is the only one that can be written simultaneously in the

Table 2
Maximized Pseudo-log-likelihood and Parameter Estimates, with Their Standard Errors, Associated with Four Archimedean Copula Models

Family	Pseudo-Log-Likelihood	$\hat{\alpha}$ (s. e.)	$\hat{\beta}$ (s. e.)	$\hat{\gamma}$ (s. e.)
Clayton	93.8	0.52 (0.032)	—	—
Frank	172.5	−3.10 (0.57)	—	—
Gumbel	207.0	1.44 (0.032)	—	—
Three-Parameter	210.1	0.138 (0.147)	1.55 (0.92)	0.9986 (0.0025)

forms (1) and (4) (Genest and Rivest 1989). Additional parametric families of maximum extreme value copulas are given by Tawn (1988) and Anderson and Nadarajah (1993), among others.

The terminology for model (4) can be justified as follows. Let $(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})$ be a random sample from bivariate distribution H, define $M_{in} = \max(X_{i1}, \ldots, X_{in})$, $i = 1, 2$, and suppose that there exist constants $a_{in} > 0$ and b_{in} for which the pair

$$\left(\frac{M_{1n} - b_{1n}}{a_{1n}}, \frac{M_{2n} - b_{2n}}{a_{2n}}\right)$$

has a non-degenerate, joint limiting distribution H^*. The marginal distributions of H^* then belong to location-scale families based either on the "extreme value" distribution $[\exp(-e^{-x}), -\infty < x < \infty]$, the Fréchet distribution $[\exp(-x^{-\alpha}), x > 0, \alpha > 0]$, or the Weibull distribution $[\exp\{-(-x)^{\alpha}\}, x < 0, \alpha > 0]$ (see Galambos 1987 for details). What may be less familiar, however, is the result of Pickands (1981) to the effect that the underlying copula associated with H^* is necessarily of the form (4). Given the long-standing concern of actuaries for the prediction of extreme events and their related costs, maximum extreme value copulas should be of considerable appeal in this area, where they would be expected to arise rather naturally. For a data-oriented presentation of univariate extreme value theory, with applications to insurance and other fields, see the recent book by Beirlant, Teugels and Vynckier (1996).

Of course, copulas in the class (4) can provide an appropriate model of dependence between variables, whether the marginals are of one of the above three types or not. The appropriateness of a family of extreme value copulas can be determined without knowledge of the marginals—as it should—using the procedure recently developed by Ghoudi, Khoudraji and Rivest (1998). Their test is based on the fact that if (X_1, X_2) is an observation from H^*, the distribution function K of $V = H^*(X_1, X_2)$ has the form $K(v) = v - (1 - \tau)v \log(v)$ for $0 \leq v \leq 1$, where

$$= \int_0^1 \frac{t(1-t)}{A(t)}\, dA'(t)$$

is the population value of Kendall's tau. Then $\mu = 8E(V) - 9E(V^2) - 1 = 0$, and since the moments of V are easy to estimate, a test statistic, Z, can be defined that rejects model (4) if the sample version of μ, divided by a jackknife estimate of its standard deviation, is significantly different from zero, as compared to the standard normal law.

value of $z = 0.06$ that is clearly insufficient to reject the maximum extreme value copula model. Because the procedure is consistent for Archimedean alternatives, this provides additional evidence that Clayton's and Frank's families are inferior to Gumbel's in this case. This is not to say that the latter family is best within the class of extreme value copulas, however.

Since dependence functions are univariate, the search for the best copula model of the form (4) can proceed along similar lines as for Archimedean copulas. To choose between various parametric families of dependence functions A_γ that have been fitted to the data using the pseudo-likelihood approach of Genest, Ghoudi and Rivest (1995), one may plot the $A_{\hat\gamma}$'s against a nonparametric estimator A_n. Tawn (1988) suggests that the classical estimator of Pickands (1981) or the variant due to Deheuvels (1991) can be used for this purpose, but a much more efficient procedure has since been proposed by Capéraà, Fougères and Genest (1997).

Alternatively, the following proposition provides ways of embedding specific parametric families of dependence functions into larger systems, so that formal or informal selection procedures may be based as before on the corresponding maximized pseudo-log-likelihoods.

Proposition 3

Let A and B be two dependence functions.

(i) (*convex combination*) If $0 \leq \lambda \leq 1$, then $\lambda A + (1 - \lambda)B$ is a dependence function.

(ii) (*asymmetrization*) If $0 < \kappa, \lambda < 1$, and if $\bar{u}$ denotes $1 - u$ for arbitrary $0 \leq u \leq 1$, the following formula defines a dependence function:

$$E(t) = (\kappa t + \lambda \bar{t})A\left(\frac{\kappa t}{\kappa t + \lambda \bar{t}}\right) + (\bar{\kappa} t + \bar{\lambda}\bar{t})B\left(\frac{\bar{\kappa} t}{\bar{\kappa} t + \bar{\lambda}\bar{t}}\right), \quad 0 \leq t \leq 1$$

As with the previous propositions, the above list is not exhaustive. Rule (i), which is mentioned by Tawn (1988), is the special case of (ii) corresponding to $\kappa = \lambda$. Function (5) is the dependence function of the extreme value copula defined by

$$C_A(u^{1-\kappa}, v^{1-\lambda})C_B(u^{\kappa}, v^{\lambda})$$

for all $0 \leq u, v \leq 1$. Rule (ii) is thus a consequence of the general asymmetrization process alluded to below the statement of Proposition 2.

To illustrate this final point, suppose that Gumbel's model, C_B, is enlarged via (6) using the independence copula $C_A(u, v) = uv$, generated by $A \equiv 1$. The resulting asymmetric model, already investigated at the end of the previous section, is then a three-parameter, maximum extreme value family of copulas, whose dependence functions are of the form

$$E(t) = \{\kappa t + \lambda(1-t)\} + \{\kappa^\alpha t^\alpha + \lambda^\alpha(1-t)^\alpha\}^{1/\alpha}, \quad 0 \leq t \quad 1.$$

Those who are familiar with the literature on multivariate extreme value theory will have recognized the class of dependence functions of what Tawn (1988) or Smith, Tawn and Yuen (1990) refer to as the asymmetric logistic model.

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